MATH 3113-008 Test I

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February 4, 2015 2:30pm-3:50pm

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____________________

Problem	Score
1	
2	
3	
4	
5	
Total	

1. (20 points) The table below lists values for the continuous function $f(x, y)$.

$y \setminus x$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
3	0	1	2	3	4	5	6
2	-1	0	1	2	3	4	5
1	-2	-1	0	1	2	3	4
0	-3	-2	-1	0	1	2	3
-1	-4	-3	-2	-1	0	1	2
-2	-5	-4	-3	-2	-1	0	1
-3	-6	-5	-4	-3	-2	-1	0

Sketch the slope field of the differential equation $y'(x) = f(x, y)$ and the solution curve of the solution with initial condition $x = 2, y = -1$.

Solution: You can use the java applet to see what the curve should look like. Use $f(x, y) = y + x$.

2. (20 points) Find the particular solution to the initial value problem

$$\frac{dy}{dx} = ye^{2x}, y(0) = 2e.$$

Solution: This is a separable equation, so we can write

$$\begin{aligned} \frac{1}{y}dy &= e^{2x}dx \\ \int \frac{1}{y}dy &= \int e^{2x}dx \\ \ln|y| &= \frac{e^{2x}}{2} + C \\ |y| &= e^{\frac{e^{2x}}{2}+C}. \end{aligned}$$

Writing $K = \pm e^C$, we have then

$$y = Ke^{\frac{e^{2x}}{2}}.$$

The initial value is $y = 2, x = 0$. Plugging those values in we obtain

$$2 = Ke^{\frac{1}{2}},$$

or in other words, $K = 2/\sqrt{e}$.

The particular solution that satisfies this initial value problem is then

$$y = (2/\sqrt{e})e^{\frac{e^{2x}}{2}}.$$

3. (25 points) In the following two questions, $x(t)$ represents the amount of salt or pollutant inside the tank after t seconds. For each question, write down a differential equation that models $x(t)$. Please do not attempt to solve the differential equations.

   (a) The tank initially contains 3 lb of salt and 5 gallons of water. Pure water enters the tank at a rate of 10 gallons per second. Assume that the solution is perfectly mixed while inside the tank. The mixture flows out of the tank at a rate of 9 gallons per second.

   (b) The tank initially contains just 3 gallons of pure water. 5 gallons of waste water enters the tank every second. This waste water has a pollutant concentration of 7%. Assume that the contents of the tank are perfectly mixed. The mixture flows out of the tank at a rate of 5 gallons per second.

Solution:

(a) No salt enters the tank, so we have $x'(t) = 0 - 9x(t)/v(t)$, where $v(t)$ is the volume of liquid in the tank at time t. We have $v(t) = 5 + 10t - 9t = 5 + t$, so

$$x'(t) = -\frac{9x(t)}{5+t}.$$

(b) 5 gallons enters the tank with concentration 7/100, and 5 gallons leave the tank with concentration $x(t)/v(t)$. Here $v(t)$ is always 3 gallons, so

$$x'(t) = 5\frac{7}{100} - 5\frac{x(t)}{3}.$$

4. (25 points) Find the general solution for $y' = (x + y + 3)^2$. Hint: the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$.

Solution: We rewrite the equation as

$$\frac{dy}{dx} = (x + y + 3)^2.$$

We make the substitution

$$u = x + y + 3.$$

Differentiating this equation with respect to x we obtain

$$\frac{du}{dx} = 1 + \frac{dy}{dx}.$$

Equivalently, of course,

$$\frac{dy}{dx} = \frac{du}{dx} - 1.$$

Substituting all these back to the original equation to eliminate the y term, we have

$$\begin{aligned} \frac{du}{dx} - 1 &= u^2 \\ \frac{du}{dx} &= u^2 + 1 \\ \int \frac{1}{u^2+1} &= \int 1 dx \\ \arctan(u) &= x + C \\ u &= \tan(x + C) \\ x + y + 3 &= \tan(x + C) \end{aligned}$$

$$y = \tan(x + C) - 3 - x.$$

5. (10 points) Demonstrate that the statement

The growth rate of x is porportional to x,

implies that x can be modelled by the growth-decay equation,

$$x(t) = Ce^{kt},$$

where C and k are constants.

Solution: "The growth rate of x is porportional to x" means that we can write

$$\frac{\left(\dfrac{dx}{dt}\right)}{x} = k,$$

for some constant k. This is a separable equation, which we then rewrite as

$$\begin{aligned} \int \frac{1}{x} dx =& \int k dt \\ \ln(|x|) =& kt + K \\ |x| =& e^{kt+K} \\ x =& Ce^{kt}, C = \pm e^{K}. \end{aligned}$$

Note: students will not be penalized for omitting the absolute value symbols $|\ldots|$, because in these growth-decay problems $x(t)$ is usually always non-negative.