MATH 4163-002 Practice Test I

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February 16, 2015 1:30pm-2:45pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
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6	
7	
Total	

1. (10 points) The conservation of energy equation for a 1-dimensional rod is written as

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial \Phi(x,t)}{\partial x}.$$

Where $e(x,t)$ is the energy density and $\Phi(x,t)$ is the flux, the rate at which heat energy flows to the right. Explain the minus sign in this equation.

Solution:
If $\frac{\partial \Phi(x,t)}{\partial x}$ is positive, this implies that the rate of heat flow leaving the right side of the rod is faster than the rate of heat flow entering the left side of the rod (alternatively, it could also mean that the incoming flow in the right side of the rod is slower than the heat flow leaving the rod on the left, but the result is the same). This implies that the rod is losing net heat energy, and so the energy density of the rod is dropping over time. Thus $\frac{\partial e(x,t)}{\partial t}$ must be negative.

By similar logic, if $\frac{\partial \Phi(x,t)}{\partial x}$ is negative this implies that the rate of heat flow leaving the right side of the rod is slower than the rate of heat flow entering the left side of the rod (alternatively, it could also mean that the incoming flow in the right side of the rod is faster than the heat flow leaving the rod on the left, but the result is the same). This implies that the rod is gaining net heat energy, and so the energy density of the rod is increasing over time. Thus $\frac{\partial e(x,t)}{\partial t}$ must be positive.

2. (15 points) Derive the heat equation for a 1-dimensional rod,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},$$

Using Fourier's law and the conservation of energy equation given in Problem 2. Here $u(x,t)$ is the temperature of the rod at location x and time t.

Solution: Fourier's law is

$$\Phi(x,t) = -K\frac{\partial u(x,t)}{\partial x},$$

where Φ is heat flux.

Note to students: Fourier's law was never provided in the practice test. You are expected to know both Fourier's law, the energy conservation equation, and the relationship between u and e given below.

Temperature and energy density are related as follows:

$$c\rho u(x,t) = e(x,t),$$

where c is specific heat and ρ is mass density. Plugging both of those in the energy conservation formula we get

$$\begin{aligned}
\frac{\partial e(x,t)}{\partial t} &= -\frac{\partial \Phi(x,t)}{\partial x} \\
\frac{\partial c\rho u(x,t)}{\partial t} &= -\frac{\partial\left(-K\frac{\partial u(x,t)}{\partial x}\right)}{\partial x} \\
\frac{\partial u(x,t)}{\partial t} &= \frac{K}{c\rho}\frac{\partial^2 u(x,t)}{\partial x^2},
\end{aligned}$$

and so once we define $k = K/c\rho$ we are done.

3. (15 points) Determine if the following two operators are linear operators or not. Justify your answers.

   (a) $A(f(x)) = \int_0^x f(t)dt$

   (b) $B(f(x)) = f(x) - 1$.

Solution:

(a)

$$\begin{aligned} A(f(x) \pm g(x)) =& \int_0^x f(t) \pm g(t)dt \\ =& \int_0^x f(t)dt \pm \int_0^x g(t)dt \\ =& A(f(x)) \pm A(g(x)). \end{aligned}$$

Also, for any constant α

$$\begin{aligned} A(\alpha f(x)) =& \int_0^x \alpha f(t)dt \\ =& \alpha \int_0^x f(t)dt \\ =& \alpha A(f(x)). \end{aligned}$$

Hence A is linear.

(b) We have $B(2f(x)) = 2f(x) + 1$, but $2B(f(x)) = 2f(x) + 2$. Thus it is not true that $B(2f(x)) = 2B(f(x))$, and so B is not linear.

4. (15 points) Consider the heat equation,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},$$

with boundary conditions $u(0,t) = u(L,t) = 0$.

The first step to solving this problem is to assume that the solution $u(x,t)$ can be written in the form

$$u(x,t) = F(x)G(t).$$

This will reduce our partial differential equation into two ordinary differential equations. What are the two ordinary differential equations? Explain in detail how to obtain them from the PDE. You **do not** have to solve the two ordinary differential equations. (Hint: if one of your ODEs is a boundary value problem, don't forget to include the boundary conditions).

Solution: Plugging in $u(x,t) = F(x)G(t)$ into our solution we obtain

$$\frac{\partial F(x)G(t)}{\partial t} = k\frac{\partial^2 F(x)G(t)}{\partial x^2},$$

which is equivalent to

$$F(x)\frac{\partial G(t)}{\partial t} = kG(t)\frac{\partial^2 F(x)}{\partial x^2}.$$

We can then move all the x-terms to the right, and all the t-terms on the left.

$$\frac{1}{kG(t)}\frac{\partial G(t)}{\partial t} = \frac{1}{F(x)}\frac{\partial^2 F(x)}{\partial x^2}.$$

But then the LHS doesn't depend on x, and the RHS doesn't depend on t. They are equal, so they both must not depend on x or t. In other words, both sides are constant. We can label this constant as $-\lambda$ and get

$$\frac{1}{kG(t)}\frac{\partial G(t)}{\partial t} = \frac{1}{F(x)}\frac{\partial^2 F(x)}{\partial x^2} = -\lambda$$

But this gets us two ordinary differential equations,

$$\frac{\partial G(t)}{\partial t} = -\lambda k G(t), \text{ and } \frac{\partial^2 F(x)}{\partial x^2} = -\lambda F(x).$$

Note that we have boundary conditions $u(0,t) = u(L,t) = 0$. This implies $F(0)G(t) = F(L)G(t) = 0$. Thus either $G(t)$ is zero, or $F(0), F(L)$ are both 0. If $G(t)$ is zero we just get the trivial solution $u(x,t) = 0$, so we may assume instead that $F(0) = F(L) = 0$. These will be our boundary conditions for our second ODE.

5. (15 points) By Fourier's Theorem, a piecewise smooth function $f(x)$ on $[0, L]$ can be written as a sum of sines in the form

$$f(x) = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{L}\right).$$

Explain how to calculate the constants B_n. You may assume the following trigonometric fact:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 \text{ whenever } m \neq n. \\ \frac{L}{2} \text{ whenever } m = n. \end{cases}$$

Solution: We multiply both sides of the equation by $\sin \frac{n\pi x}{L}$, and then integrate from $x = 0$ to $x = L$.

$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \sum_{m=1}^{\infty} \int_0^L B_m \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Based on the "trigonometric fact", the RHS is zero whenever $n \neq m$. So we have

$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{LB_n}{2},$$

and so

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

6. (15 points) Using the maximum principle, explain why there can be only one steady state solution for the heat equation for a 2-dimensional disk of radius 1.

Solution:

Assume that there are two steady-state solutions, $u(r,\theta), v(r,\theta)$ with the same boundary condition. Then we write $w(r,\theta) = u(r,\theta) - v(r,\theta)$. Since the heat equation is a homogeneous linear equation, subtracting two solutions of the heat equation will still get us a solution of the heat equation. It is also easy to check that $w(r,\theta)$ is steady-state if $u(r,\theta)$ and $v(r,\theta)$ are steady state. Thus $w(r,\theta)$ is a steady-state solution of the heat equation- and we get its boundary values by subtracting the boundary values of $u(r,\theta)$ with the boundary values of $v(r,\theta)$, so $w(r,\theta) = 0$ everywhere on the boundary of the disk.

But the maximum principle says that the maximum and minimum values of $w(r,\theta)$ are on the boundary and so we know that both the maximum and minimum values of $w(r,\theta)$ must be 0, and hence $w(r,\theta) = 0$ everywhere. This implies that

$$u(r,\theta) - v(r,\theta) = 0$$

everywhere, and so $u(r,\theta) = v(r,\theta)$.

7. (15 points) Consider the boundary value problem given by

$$F''(x) + \lambda F(x) = 0,$$

with initial conditions $\frac{d}{dx}F(0) = \frac{d}{dx}F(L) = 0.$

(a) Is 0 an eigenvalue for this problem? If it is, find the corresponding eigenfunction.

(b) Show that this boundary value problem has no negative eigenvalues.

Solution on next page

Solution:

(a) We plug in $\lambda = 0$, which gets us $F''(x) = 0$. Integrating twice, we have $F(x) = Ax + B$, for constants A, B. We plug in in $F(0) = 0$ to get $A = 0$. We plug in $F'(L) = 0$ and also get $A = 0$. Thus $F(x) = B$ is a solution for any constant B, and so $\lambda = 0$ is an eigenvalue.

(b) Let us write $\lambda = -\alpha^2$ for $\alpha > 0$. When we plug this is to the equation, we get

$$F''(x) - \alpha^2 F(x) = 0,$$

Using $F(x) = e^{rx}$ we get

$$r^2 - \alpha^2 = 0,$$

and so $r = \pm\alpha$. This implies that our general solution is

$$F(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}.$$

Using the boundary condition $F'(0) = 0$ we get that

$$0 = \alpha C_1 - \alpha C_2,$$

and so $C_2 = C_1$ and

$$F(x) = C_1 e^{\alpha x} + C_1 e^{-\alpha x} = C_1(e^{\alpha x} + e^{-\alpha x}).$$

Plugging in the other boundary condition gets us

$$0 = C_1(\alpha e^{\alpha L} - \alpha e^{-\alpha L}).$$

To get a nonzero solution, we need C_1 to be nonzero. But this is only possible if

$$e^{\alpha L} = e^{-\alpha L}.$$

Taking logs of both sides, this implies $\alpha L = -\alpha L$, which is impossible since $\alpha > 0$. Thus $C_1 = 0$, and so $C_2 = 0$, and $F(x) = 0$ is our only solution.

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Grading Key Please note that this grading key is just a rough plan of how I plan to grade this test. I reserve the right to change the grading scale once I am actually grading the test.

1. A grades are for answers that clearly indicate the student understands the role of the minus sign by explaining the physical intuition of the equation correctly. B grades are for vague, non-specific answers that nevertheless suggest the student understands the equation. For example, the student might explain correctly how $e(x,t)$ and $\Phi(x,t)$ are related without directly (or correctly) addressing the minus sign. C grades are for students who know what $e(x,t)$ and $\Phi(x,t)$ are, and can more or less correctly explain what their derivatives mean, but make no furhter progress in answering the question. D grades are for students who provide a vague or slightly incorrect description of what $e(x,t), \Phi(x,t)$ and their derivatives are.

2. A grades are for correct answers, perhaps with inconsequential errors in calculation or manipulating constants. B grades are for answers that are mostly correct, but are perhaps missing a step. C grades are for a correct list of all the equations that lead to the heat equation but without a description of how to put them together. D grades are for a list of equations that lead to the heat equation, but written in a manner that is either incorrect or incomplete.

3. A grades are for correct answers to both parts, including correct justifications (perhaps with minor calculation errors). B grades are for answers that demonstrate that the student knows the difference between linear and non-linear operators and knows how to start checking for linearity, but somehow have gotten either one or both of the parts of the problem wrong. C grades are for correct descriptions of the difference between linear and non-linear operators, but with insignificant progress towards checking if the given operators are linear . D grades are for definitions of linearity that are only partially correct.

4. A grades are for correct answers, perhaps with minor calculation errors. B grades are for answers that are more or less correct, except with one of the major steps missing or wrong. C grades are for answers that are partially correct, but with multiple major steps missing or wrong. D grades are for answers that have managed to get at least one step right.

5. A grades are for correct answers, perhaps with minor calculation errors. B grades have serious calculation errors that call into question whether the student understands the derivation. C grades are for answers that start the problem correctly, but fail to use the hint properly. D grades are for answers that make some progress, but which get stuck before the hint even comes into play.

6. A grades are for correct answers, perhaps with minor omissions or calculation errors. B grades are for answers that miss or mess up important steps, but are more or less correct. C grades are for answers that include a correct definition of the maximum principle, and at least a vague description as to how to start solving the problem. D

grades contain at least partially correct descriptions of the maximum principle, without progress toward showing there is a unique steady-state solution.

7. A grades are correct answers, perhaps with a minor mistake or omission. B grades are answers that are wrong, but nevertheless demonstrate that the student understand what eigenvalues are and how to look for them. C grades are answers that are wrong, but which start the problem correctly. D grades are for answers that demonstrate ta knowledge of what eigenvalues and eigenfunctions are, but makes no further progress.