MATH 3113-002 Test I

Dr. Darren Ong

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Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points) (a) Calculate the slopes for solutions of the differential equation $y' = x - 2y$. You should use the table below

$y \setminus x$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
3							
2							
1							
0							
-1							
-2							
-3							

(b) Sketch the slope field and sketch the solution of the differential equation $y' = x - 2y$ with initial condition $y(2) = -2$.

Solution:

$y \setminus x$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
3	-9	-8	-7	-6	-5	-4	-3
2	-7	-6	-5	-4	-3	-2	-1
1	-5	-4	-3	-2	-1	0	1
0	-3	-2	-1	0	1	2	3
-1	-1	0	1	2	3	4	5
-2	1	2	3	4	5	6	7
-3	3	4	5	6	7	8	9

As for part (b), embedding images in pdf files is a pain, so just use the DFIELD java applet and look at it yourself!

2. (20 points) Consider the differential equation $y' = \sin(x) + 1$

   (a) Calculate the general solution of this differential equation.

   (b) Assume that in addition, the solution of the differential equation satisfies $y(0) = 3$. Calculate $y(x)$.

Solution:

(a) We integrate both sides to obtain

$$y = \int \sin(x) + 1 dx = -\cos(x) + x + C.$$

(b) We plug in $x = 0$, $y = 3$ in the above equation and find

$$3 = -\cos(0) + 0 + C,$$

and therefore $C = 4$. Thus our particular solution must be $y(x) = -\cos(x) + x + 4$.

3. (20 points) The half-life of unobtainium is $\ln(2)$ years (roughly 0.69 years). An accident in an unobtainium reactor on Pandora causes the levels of unobtainium radiation on the planet to be e^3 times (roughly 20.09 times) the acceptable level for human habitation. How long before radiation levels on Pandora become tolerable for humans? *Note: I have carefully chosen values so a calculator is unecessary to solve this problem.*

Solution: Let $x(t)$ be the radiation levels on Pandora t years after the accident. Let the unit of $x(t)$ be the level acceptable for human habitation, so our goal is to find T so $x(T) = 1$.

We know $x(t)$ is modeled by the growth-decay equation, so

$$x(t) = Ce^{kt},$$

for constants C and k.

We know the radiation levels is e^3 at the time of the accident, so $x(0) = e^3$. This implies $e^3 = Ce^{k0} = C$, and so

$$x(t) = e^3 e^{kt}.$$

Since the half life of unobtainium is $\ln(2)$ years, $x(\ln(2))$ must equal $e^3/2$. This implies

$$\frac{e^3}{2} = x(\ln(2)) = e^3 e^{k\ln(2)} = e^3 2^k.$$

In other words, $\frac{1}{2} = 2^k$, so $k = -1$. We then know that

$$x(t) = e^3 e^{-t} = e^{3-t}.$$

Clearly, $x(T) = 1$ only when $T = 3$. So it takes 3 years before the radiation levels are acceptable for human habitation.

4. (20 points) Calculate the particular solution for the following initial value problem.

$$y' + \frac{y}{x} = x, y(1) = 2$$

Solution: The integrating factor is

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

(We may assume x is positive due to the initial condition, so we don't need to write $\ln|x|$.)

Multiplying the equation by the integrating factor we get

$$\frac{dy}{dx}x + y = x^2.$$

This simplifies to

$$\frac{d}{dx}(yx) = x^2.$$

Integrating both sides we get

$$yx = \frac{x^3}{3} + C,$$

and we conclude that the general solution is $y = x^2/3 + C/x$.

Plugging in the initial condition $x = 1, y = 2$ we find that

$$2 = \frac{(1)^2}{3} + \frac{C}{1}.$$

So solving for C, we find that $C = 4/3$. Thus the solution to the initial value problem is

$$y = \frac{x^2}{3} + \frac{4}{3}.$$

5. (10 points) Consider the following homogeneous equation in variables y, x:

$$y'yx + x^2 = y^2$$

Given the substitution $u = y/x$, rewrite the above equation as a simpler differential equation in variables u, x. Please don't attempt to solve this equation.

Solution: We start by dividing both sides by the highest power of x, that is x^2. We get

$$\frac{dy}{dx}\frac{y}{x} + 1 = \left(\frac{y}{x}\right)^2$$

Since we are using the substitution $u = y/x$, we also have $y = ux$. Differentiating both sides of this equation, we have

$$\frac{dy}{dx} = \frac{du}{dx}x + u.$$

This enables us to rewrite our differential equation as

$$\left(\frac{du}{dx}x + u\right)u + 1 = u^2.$$

6. (10 points) Nissa wants to find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x-1}.$$

She recognizes that this is a separable equation, and so she decides to rearrange terms as follows:

$$\int \frac{1}{y} dy = \int \frac{1}{x-1} dx.$$

Integrating both sides, she finds that

$$\ln|y| = \ln|x-1| + C,$$

for some arbitrary constant C.

Nissa then exponentiates both sides of the equation.

$$|y| = |x-1|e^C.$$

Dropping the absolute values, she then gets

$$y = \pm(x-1)e^C.$$

She redefines the constant as $K = \pm e^C$ to get this expression for the general solution:

$$y = K(x-1).$$

However, when Nissa tries to plug in the initial condition $y(1) = 3$, it is impossible for her to find an appropriate value for the constant K. What (if anything) has gone wrong with her calculation?

Solution: The differential equation fails the existence test, since $\frac{y}{x-1}$ is not continuous around the point $(1, 3)$. Thus we cannot guarantee a solution with initial condition $x = 1, y = 3$.