Math 101 Spring Session Midterm II

Instructions: This is a **80**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. (21 points total, 3 points each) Find the *general* antiderivative $F(x)$ for the following functions $f(x)$.

a) $f(x) = 0$

$F(x) = C$

b) $f(x) = \pi$

$F(x) = \pi x + C$

c) $f(x) = x^3 + 7x$

$F(x) = \frac{x^4}{4} + \frac{7x^2}{2} + C$

d) $f(x) = -\frac{2}{x}$

$F(x) = -2\ln(x) + C$

e) $f(x) = \sin(x)$

$F(x) = -\cos(x) + C$

f) $f(x) = -e^x$

$F(x) = -e^x$

g) $f(x) = \frac{1}{\sqrt[3]{x}}$

$f(x) = x^{-1/3}$

$F(x) = \frac{x^{2/3}}{2/3}$

2. (10 points) At time $t \geq 0$, the radius of a circle is growing at a rate of e^{2t} centimeters a second. Calculate how quickly the circle's area is growing when its radius is 10cm. $t=1$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

I don't care about units

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot \cancel{\frac{dr}{dt}}\; e^{2t}$$

$$= 20\pi e^{2t}$$

$$= 20\pi e^{2}$$

3. (15 points total, 5 points each) Calculate the following limits.

(a)

$$\lim_{x\to\infty}\frac{x^2}{e^{-x}} \sim \frac{\infty}{0}$$

~~$\frac{\infty}{\infty}$, so L'Hopital~~

$$= \infty$$

(b)

$$\lim_{x\to\infty}\frac{\ln(x)}{x} = \lim_{x\to\infty}\frac{1/x}{1}$$

$\frac{\infty}{\infty}$, so L'Hopital

$$= \frac{0}{1} = 0$$

(c)

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x}$$

$\frac{0}{0}$, so L'Hopital

$$= \lim_{x\to 0}\frac{1}{2}\,\frac{\sin x}{x}$$

$$= \frac{1}{2}$$ (basic trig limit)

4. (9 points) Show that the equation $1+2x+x^3+4x^5 \stackrel{?}{=} 0$ has exactly one real root. (There will be considerable partial credit even if you can only show that it has at least one real root)

At least one real root:

Note that $f(x) = 1+2x+x^3+4x^5$ continuous

$f(-1) = -6$, $f(0) = 1$

so by Intermediate Value Theorem there must exist a root in $(-1, 0)$.

At most one real root

$f'(x) = 2 + 3x^2 + 20x^4$

> 0 for all x

Function is strictly increasing, so can only cross the x-axis at most once.

Alternative proof:

$f'(x) > 0$ for all x

So if there were two distinct roots r_1, r_2, $f(r_1) = f(r_2) = 0$,

Then by Rolle's Theorem there must be a $c \in (r_1, r_2)$ so $f'(c) = 0$. But this is not possible.

5. (20 points) On the next page, make a detailed sketch of the graph $y = xe^{-x}$. Make sure that you clearly note

(a) all inflection points, and all local maxima and local minima (state the x and y-coordinates of all inflection points and critical points).

(b) the intervals for which the graph is increasing, decreasing, concave up and concave down.

(c) the asymptotes of the graph, if there are any.

Increasing / Decreasing $\quad f(x) = xe^{-x}$

$$f'(x) = e^{-x} - xe^{-x}$$

$\therefore$ crit points when $f'(x) = 0$

$$e^{-x} - xe^{-x} = 0$$

$$(1-x)e^{-x} = 0$$

$x = 1$ only possible critical point.

$y = e^{-1}$ $\quad (1, e^{-1})$ is a critical point

$\oplus \quad | \quad \ominus$

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$$\boxed{\begin{array}{l} f \text{ increasing at } (-\infty, 1), \text{ decreasing at } (1, \infty) \\ \text{local max at } (1, e^{-1}). \end{array}}$$

$$f''(x) = -e^{-x} - (1-x)e^{-x}$$

$f''(x) = 0 \Rightarrow -e^{-x} - (1-x)e^{-x} = 0$

$$e^{-x}(-2+x) = 0$$

$\therefore$ $x=2$ has possible inflection point.

$(2, 2e^{-2})$ has possible inflection point.

$\ominus \quad | \quad \oplus$

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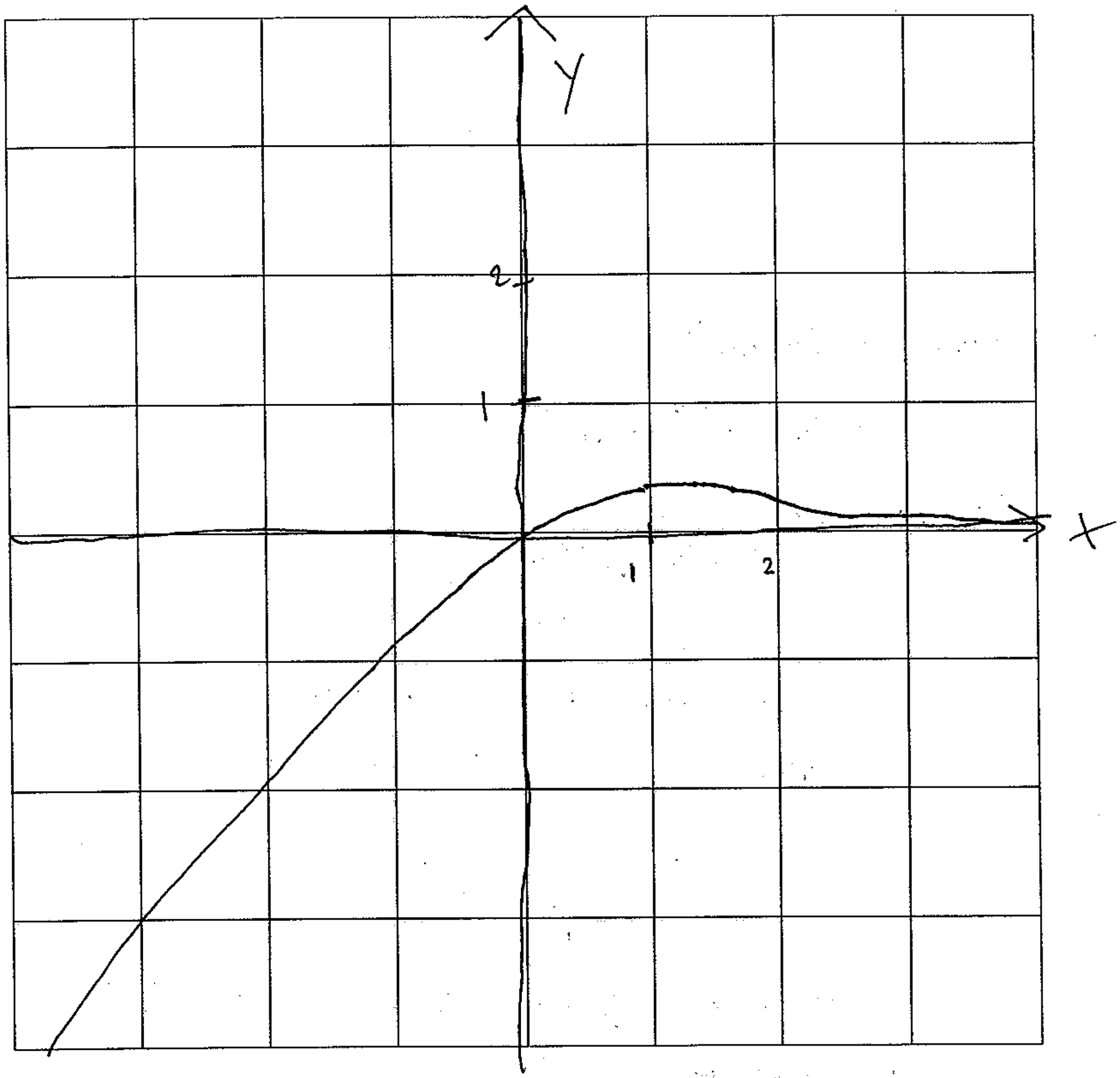
$$\boxed{\begin{array}{l} f \text{ concave down at } (-\infty, 2), \text{ concave up at } (2, \infty) \\ \text{inflection point at } (2, 2e^{-2}) \end{array}}$$



Asymptotes

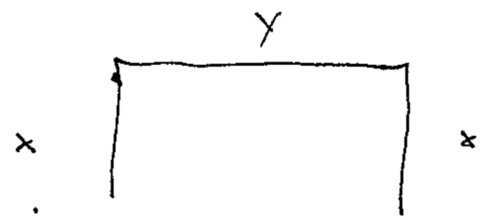
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{\frac{\infty}{\infty}\ \text{L'H}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} xe^{-x} = \lim_{x\to-\infty} \frac{x}{e^{x}} = -\infty$$

$\therefore$ Horizontal asymptote ~~at~~ x at 0 to the right.

6. (20 points) The RDA Corporation wants you to design a military base on Pandora. The base will sit on a rectangular compound that has area 16 square miles, and the boundaries of the base will run north to south, east to west. You will have to build concrete walls on the north, west, and east boundaries of the compound, in order to fend off attacks from the native Na'vi tribes. It will not be necessary to fortify the south boundary of the base.

What are the dimensions of the compound that minimizes the length of the concrete walls you will need to build?



$xy = 16 \qquad y = \frac{16}{x}$

minimize $2x + y$

$F(x) = 2x + y$

$= 2x + \frac{16}{x}$

$F'(x) = 2 - \frac{16}{x^2}$

$F'(x) = 0 \Rightarrow 2 - \frac{16}{x^2} = 0$

$x^2 = 8$

$x = \pm\sqrt{8}$ (Discard $x = -\sqrt{8}$ answer)

$y = \frac{16}{\sqrt{8}} = 2\sqrt{8}.$

$\boxed{x = \sqrt{8}, \ y = 2\sqrt{8}}$

7. (5 points) Consider the following graph of $y = f(x)$, where $f(x)$ is a differentiable function. The value $x = x_1$ on the graph is an initial approximation to the root of $f(x)$. Illustrate pictorially how Newton's method is used to find the third approximation, x_3, and mark x_3 on the x-axis. (x_3 doesn't necessarily lie on the tick marks on the x-axis). No algebra is necessary for this problem!