Math 101 Spring Session Midterm I

Instructions: This is a **75**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points total, 4 points each) Calculate $\frac{dy}{dx}$ for the following values of y.

a) $y = \sqrt{\pi}$

$\boxed{\frac{dy}{dx} = 0}$, since $\sqrt{\pi}$ is a constant

b) $y = e^x$

$\boxed{\frac{dy}{dx} = e^x}$

c) $y = 2\cosh(x)$

$\boxed{\frac{dy}{dx} = 2\sinh(x)}$

d) $y = \arcsin(x)$

$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$

e) $y = \frac{\sin^2(x)\sqrt[4]{x+2}}{(2x^2+1)^{3/2}}$

$\ln y = 2\ln(\sin x) + \frac{1}{4}\ln(x+2) - \frac{3}{2}\ln(2x^2+1)$

Differentiate both sides with respect to x:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{\sin x}\cdot\cos x + \frac{1}{4}\cdot\frac{1}{x+2} - \frac{3}{2}\,\frac{1}{2x^2+1}\cdot 4x$$

$$\frac{dy}{dx} = y\cdot\left(\frac{2\cos x}{\sin x} + \frac{1}{4(x+2)} - \frac{6x}{2x^2+1}\right)$$

$$= \boxed{\frac{\sin^2(x)\sqrt[4]{x+2}}{(2x^2+1)^{3/2}}\left(\frac{2\cos x}{\sin x} + \frac{1}{4(x+2)} - \frac{6x}{2x^2+1}\right)}$$

2. (20 points total, 10 points each) For the following functions f, find $f'(x)$, and calculate the slope of the tangent line to the curve $y = f(x)$ at the point $(\pi, f(\pi))$.

(a) $f(x) = \arctan(\sin\left(\frac{x}{3}\right))$

$$\boxed{f'(x) = \frac{1}{1+(\sin\frac{x}{3})^2} \cdot \cos\left(\frac{x}{3}\right) \cdot \frac{1}{3}}$$

by chain rule.

$$f'(\pi) = \frac{1}{1+(\sin\frac{\pi}{3})^2} \cdot \cos\left(\frac{\pi}{3}\right) \cdot \frac{1}{3}$$

$$= \frac{1}{1+\frac{3}{4}} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

$$= \boxed{\frac{2}{21}}$$

(b) $f(x) = \frac{\ln x}{\cos x}$

Quotient rule:

$$\boxed{f'(x) = \frac{\cos(x)\frac{1}{x} - \ln(x)(-\sin(x))}{\cos^2 x}}$$

(LOW D-HIGH LESS HIGH D-LOW...)

$$f'(\pi) = \frac{(-1)\frac{1}{\pi} - \ln(\pi)(0)}{(-1)^2}$$

$$= \boxed{-\frac{1}{\pi}}$$

3. (20 points total, 5 points each) Calculate the following limits, if they exist.

(a)

$$\lim_{x\to\pi} \frac{1}{2}$$

$$= \boxed{\frac{1}{2}}$$

(b)

$$\lim_{x\to 1} \frac{1-x^2}{x-1}$$

$$= \lim_{x\to 1} \frac{(1-x)(1+x)}{x-1}$$

$$= \lim_{x\to 1} \frac{-(x-1)(1+x)}{(x-1)} = \lim_{x\to 1} -(1+x)$$

$$= \boxed{-2}$$

(c)

$$\lim_{x\to\infty} \frac{2-3x^3}{2x^3-3x+5}$$

$$= \lim_{x\to\infty} \frac{\frac{2-3x^3}{x^3}}{\frac{2x^3-3x+5}{x^3}} = \lim_{x\to\infty} \frac{\cancel{\frac{2}{x^3}}^{\to 0} - 3}{2 - \cancel{\frac{3}{x^2}}^{0} + \cancel{\frac{5}{x^3}}^{\to 0}}$$

$$= \lim_{x\to\infty} \frac{-3}{2}$$

$$\boxed{= -\frac{3}{2}}$$

(d)

$$\lim_{x\to 0} \frac{\sin(3x)}{x}$$

$$= \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot 3$$

$$= 1 \cdot 3$$

$$\boxed{= 3}$$

alternatively, note that for small x, $\sin(3x)$ is close to $3x$.

4. (20 points)The curve

$$x^3 + y^3 = 3xy$$

is known as the *Folium of Descartes*. Find the equation of the tangent line to this curve at the point $(\frac{2}{3}, \frac{4}{3})$.

Differentiate both sides with respect to x

Implicit Differentiation

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 3y + 3x\left(\frac{dy}{dx}\right)$$

$$3y^2\left(\frac{dy}{dx}\right) - 3x\left(\frac{dy}{dx}\right) = 3y - 3x^2$$

$$(3y^2 - 3x)\left(\frac{dy}{dx}\right) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

Plug in $x = \frac{2}{3}$, $y = \frac{4}{3}$

Slope of tangent line at $(\frac{2}{3}, \frac{4}{3})$ $= \dfrac{3(\frac{4}{3}) - 3(\frac{2}{3})^2}{3(\frac{4}{3})^2 - 3(\frac{2}{3})}$

$$= \frac{4 - \frac{4}{3}}{\frac{16}{3} - 2}$$

$$= \frac{8}{10} = \frac{4}{5}$$

We need to find a line with slope $\frac{4}{5}$ through the point $(\frac{2}{3}, \frac{4}{3})$

$$\frac{y - \frac{4}{3}}{x - \frac{2}{3}} = \frac{4}{5} \Rightarrow 5y - \frac{20}{3} = 4x - \frac{8}{3}$$

$$\boxed{5y - 4x - 4 = 0}$$

5. (10 points total)Is it true that, for differentiable functions $f(x), g(x)$

$$\frac{d(f(x)g(x))}{dx} = \frac{df(x)}{dx} \cdot \frac{dg(x)}{dx}?$$

If you think this formula is true, prove it using the limit definition of the derivative. If you don't think this formula is true, find differentiable functions f, g for which it does not hold.

No, it's not true.

Try $f(x) = x, \quad g(x) = x$

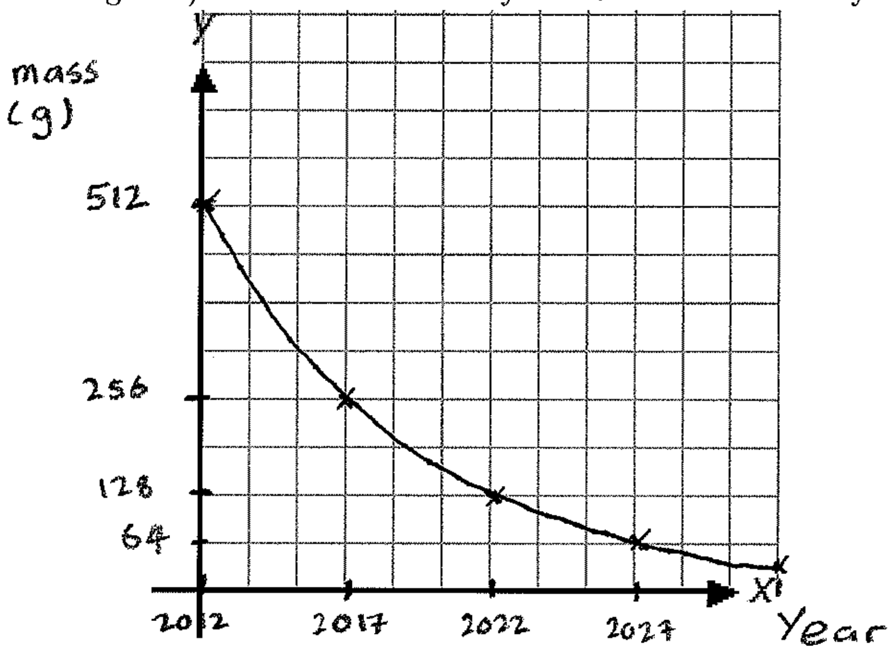
$$\frac{d\,(f(x)(g(x)))}{dx} \Rightarrow \frac{d\,x^2}{dx} = 2x$$

but $$\frac{d\,f(x)}{dx} \cdot \frac{d\,g(x)}{dx} = 1 \cdot 1 = 1$$

$$2x \neq 1$$

6. (10 points total) The *half-life* of the radioactive element Unobtanium is 5 years. In other words, half of every sample of Unobtanium decays after five years. For example, if you start with 1kg of Unobtanium, you'll be left with 0.5kg after 5 years, 0.25kg after 10 years, 0.125kg after 15 years, and so on.

   (a) (4 points) Assume that a sample consists of 512 grams of Unobtanium in the year 2012. Sketch a graph where the y-axis is the mass of the Unobtanium sample (in grams) and the x-axis is the year. Remember to label your axes!



   (b) (6 points) Write down the equation of the graph.

This is an exponential function with base $\left(\frac{1}{2}\right)$

Start with $y = \left(\frac{1}{2}\right)^x$. We need to shift 2012 units to the right

$y = \left(\frac{1}{2}\right)^{(x-2012)}$. We need to start at $y=512$ when $x=2012$

$y = 512\left(\frac{1}{2}\right)^{(x-2012)}$. We need to adjust the function so it halves every five years, instead of every year.

$$\boxed{y = 512\left(\frac{1}{2}\right)^{\left(\frac{x-2012}{5}\right)}}$$