# MATH 3113-005 Test III

Dr. Darren Ong

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| Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed. |

Name: ______________________________

| Problem | Score |
| --- | --- |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| Total | |

**Formula Sheet**

| $f(t)$ | $\mathcal{L}(f(t))$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^a (a > -1)$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |

The above Laplace transforms are valid for $s > 0$, with the exception of $\sinh(kt)$ and $\cosh(kt)$ which are valid for $s > |k|$.

$$n \text{ integer}, \Gamma(n+1) = n!$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}\left(\int_0^t f(\theta) d\theta\right) = \frac{\mathcal{L}(f(t))}{s}$$

$$F(s) = \mathcal{L}(f(t)) \implies F(s-a) = \mathcal{L}(e^{at} f(t))$$

$$f(t) * g(t) = \int_0^t f(\theta) g(t-\theta) d\theta$$

$$\mathcal{L}(-t f(t)) = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma$$

1. (10 points) Write down $\mathcal{L}(f(t))$ for the following functions $f(t)$ (no work needs to be shown for this problem).

(a) $f(t) = 0$

(b) $f(t) = \sin(\pi t)$

(c) $f(t) = \frac{1}{\sqrt{t}}$

(d) $f(t) = (e^t)^2$

(e) $f(t) = 3 - 2\sinh(2t)$

**Solution:**

(a) $0$

(b) $\pi/(s^2 + \pi^2)$

(c) $\Gamma(1/2)/\sqrt{s}$

(d) $1/(s-2)$

(e) $3/s - 4/(s^2 - 4)$

2. (10 points) Write down $\mathcal{L}^{-1}(F(s))$ for the following functions $F(s)$ (no work needs to be shown for this problem).

   (a) $F(s) = 0$

   (b) $F(s) = \frac{s+2}{s^2+4}$

   (c) $F(s) = \frac{6}{s^2-4}$

   (d) $F(s) = \frac{120}{s^6}$

   (e) $F(s) = \frac{17}{s} - \frac{1}{s-17}$

**Solution:**

(a) $0$

(b) $\cos(2t) + \sin(2t)$

(c) $3\sinh(2t)$

(d) $t^5$

(e) $17 - e^{17t}$

3. (10 points) For $x, y$ functions of $t$, write down the system of equations

$$x' = 2x + 4y + 3e^t, y' = 5x - y - t^2,$$

as a matrix equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$.

**Solution:**

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$$

4. (25 points) Consider the boundary value problem

$$y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0.$$

Find all eigenvalues $\lambda$. For each $\lambda$, find one associated eigenfunction.

**Solution:**

(i) First consider negative eigenvalues $\lambda = -\alpha^2 < 0$ for some $\alpha > 0$. Our equation is $y'' - \alpha^2 y = 0$. Substituting $y = e^{rt}$ we get that $r = \pm\alpha$, so the general solution is $y(t) = C_1 e^{\alpha} + C_2 e^{-\alpha}$. This implies

$$y'(t) = \alpha C_1 \alpha e^{\alpha t} - C_2 \alpha e^{-\alpha t}.$$

Plugging in $t = 0, \pi$ and using the given initial conditions we have $C_1 - C_2 = 0$, and $C_1 e^{\alpha\pi} - C_2 e^{\alpha\pi} = 0$. These two equations imply that $e^{\alpha\pi} = e^{-\alpha\pi}$ and so $e^{2\alpha\pi} = 1$, which is impossible for $\alpha > 0$. Thus there are no negative eigenvalues.

(ii) Next, consider the possibility that $\lambda = 0$. In this case our differential equation becomes $y'' = 0$, which has a general solution $y(t) = C_1 t + C_2$. We have $y'(t) = C_1$, and plugging in the initial conditions we have $C_1 = 0$. Thus $y(t) = C_2$ is a solution for any constant $C_2$, and we can check that it works by plugging it in. Thus $\lambda = 0$ is an eigenvalue, and $y(t) = C_2$ is a corresponding eigenfunction for any nonzero constant $C_2$.

(iii) Lastly, consider the case where $\lambda = \alpha^2 > 0$ for some $\alpha > 0$. Our differential equation becomes $y'' + \alpha^2 y = 0$. We substitute $y(t) = e^{rt}$, and we obtain $r = \pm i\alpha$. Thus our general solution is $y(t) = C_1 \cos(\alpha t) + C_2 \sin(\alpha t)$. Then,

$$y'(t) = -C_1 \alpha \sin(\alpha t) + C_2 \alpha \cos(\alpha t).$$

The $y'(0) = 0$ then implies $0 = C_2\alpha$, and so $y'(\pi) = 0$ then implies $0 = -C_1 \alpha \sin(\alpha\pi)$. Thus for $C_1$ to be nonzero, we need $\sin \alpha(\pi) = 0$. This is only true if $\alpha$ is an integer. Thus the only eigenvalues happen when $\alpha = 1, 2, 3, \ldots$, so $\lambda = 1, 4, 9, 16, \ldots$. For each $\lambda = \alpha^2$, a corresponding eigenvector is $C_1 \cos(\alpha t)$ for any nonzero constant $C_1$.

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5. (15 points) Solve the initial value problem

$$x''(t) + 16x(t) = 0; x(0) = 2, x'(0) = -3,$$

using Laplace transforms. **You will get no credit for solving this problem using any other method.**

**Solution:** Taking $F(s) = \mathcal{L}(x(t))$, we have

$$\mathcal{L}(x''(t)) = s^2 F(s) - 2s - (-3).$$

Thus taking the Laplace transforms of both sides gets us

$$(s^2 + 16)F(s) - 2s + 3 = 0.$$

And so,

$$F(s) = \frac{2s - 3}{s^2 + 16}$$

Using the table, we find that

$$x(t) = \mathcal{L}^{-1}(F(s)) = 2\cos(4t) - \frac{3}{4}\sin(4t).$$

6. (10 points) Calculate the Laplace transform

$$\mathcal{L}(e^{3t}\cosh(10t))$$

**Solution:** $\mathcal{L}(\cosh(10t)) = \frac{s}{s^2-10^2}$, so

$$\mathcal{L}(e^{3t}\cosh(10t)) = \frac{s-3}{(s-3)^2-10^2}.$$

7. (10 points) Show that the initial value problem

$$x''(t) + 9x(t) = f(t); x(0) = 0, x'(0) = 0,$$

has a solution

$$x(t) = \frac{1}{3}\int_0^t f(t-\theta)\sin(3\theta)d\theta.$$

**Solution:** Taking $F(s) = \mathcal{L}(x(t))$, we have $\mathcal{L}(x''(t)) = s^2F(s)$ and so taking the Laplace transform of both sides, we get $(s^2+9)F(s) = \mathcal{L}(f(t))$, or equivalently

$$F(s) = \frac{\mathcal{L}(f(t))}{s^2+9}.$$

Thus

$$\begin{aligned} x(t) = \mathcal{L}^{-1}F(s) =& \mathcal{L}^{-1}(\mathcal{L}(f(t))) * \mathcal{L}^{-1}(1/(s^2+9)) \\ =& f(t) * \frac{1}{3}\sin(3t) \\ =& \frac{1}{3}\int_0^t f(t-\theta)\sin(3\theta)d\theta. \end{aligned}$$

8. (10 points) Let $f(t), g(t)$ be two functions for which Laplace transforms $\mathcal{L}(f(t))$, $\mathcal{L}(g(t))$ exist for $s > c$, for some constant $c$. Assume also that $g(t)$ and $\mathcal{L}(g(t))$ are always nonzero. Is it true that

$$\mathcal{L}\left(\frac{f(t)}{g(t)}\right) = \frac{\mathcal{L}(f(t))}{\mathcal{L}(g(t))}?$$

If it is true, derive this formula using the definition of the Laplace transform. If it is not true, use a counterexample to show that the formula is false.

**Solution:** The statement is false. As a counterexample, take $f(t) = e^t$, $g(t) = 1$. The LHS becomes $1/(s-a)$, and the RHS becomes $s/(s-a)$. There are also several other possible counterexamples.