MATH 3113-005 Test II

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Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
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Total	

1. (10 points) The differential equation given by

$$3y''(t) + y'(t) - 2y(t) = 2\cos(t)$$

has a solution $y(t) = -\frac{5}{13}\cos(t) + \frac{1}{13}\sin(t)$.

On the other hand,

$$3y''(t) + y'(t) - 2y(t) = 0$$

has two linearly independent solutions $y(t) = e^t$, $y(t) = e^{-2t/3}$.

Write down (without proof) the general solution for

$$3y''(t) + y'(t) - 2y(t) = 2\cos(t).$$

Solution: The general solution is a sum of the particular solution y_p and the complementary solution y_c. We are given that

$$y_p(t) = -\frac{5}{13}\cos(t) + \frac{1}{13}\sin(t),$$

and by the principle of superposition, for arbitrary constants C_1, C_2,

$$y_c(t) = C_1 e^t + C_2 e^{-2t/3}.$$

Thus

$$y(t) = -\frac{5}{13}\cos(t) + \frac{1}{13}\sin(t) + C_1 e^t + C_2 e^{-2t/3}.$$

2. (15 points) Show that if a, b are real numbers and $a \neq b$, then e^{at} and e^{bt} are linearly independent.

Solution: The Wronskian of e^a, e^b is

$$
\begin{aligned}
W =& e^{at}(e^{bt})' - (e^{at})'e^{bt} \\
=& be^{at}e^{bt} - ae^{at}e^{bt} \\
=& (b-a)e^{at}e^{bt}
\end{aligned}
$$

Since $a \neq b$, $b - a \neq 0$, and e^{at}, e^{bt} are both nonzero since they are exponential. Thus $W \neq 0$ and so e^{at}, e^{bt} are linearly independent.

3. (15 points) Where y is a function of x, solve the initial value problem given by

$$y''(x) - 4y'(x) + 3y(x) = 0,$$

with initial conditions $y(0) = 7, y'(0) = 11$.

Solution: We make the substitution $y = e^{rx}$, and transform the equation to

$$e^{rx}(r^2 - 4r + 3) = 0$$

Since exponentials are nonzero, we have to find the roots of the quadratic factor. Using the quadratic formula we have

$$r = \frac{4 \pm \sqrt{16 - 12}}{2},$$

or $r = 1, 3$. This implies that $y(x) = e^x, y(x) = e^{3x}$ are linearly independent solutions (by Problem 2) and so the general solution is given by

$$y(x) = C_1 e^x + C_2 e^{3x}.$$

We take the derivative to get

$$y'(x) = C_1 e^x + 3C_2 e^{3x}.$$

The initial condition $y(0) = 7$ gets us $7 = C_1 + C_2$, and the initial condition $y'(0) = 11$ gets us $11 = C_1 + 3C_2$. Subtracting the second equation from the first gets us $2C_2 = 4$, so $C_2 = 2$. It is then clear that $C_1 = 5$.

This implies that the solution to the IVP is

$$y(x) = 5e^x + 2e^{3x}.$$

4. (10 points) Calculate the general solution of the differential equation

$$y''(x) + 6y'(x) + 9y(x) = 0.$$

You may assume that pairs of functions of the form $\{e^{kx}, xe^{kx}\}$ or $\{e^{ax}\sin(bx), e^{ax}\cos(bx)\}$ are linearly independent.

Solution: Making the substitution $y(x) = e^{rx}$, we reduce the problem to

$$r^2 + 6r + 9 = 0.$$

Using the quadratic formula, we find that this quadratic equation factors to

$$(r + 2)^2 = 0,$$

and so we have a double root at $r = -2$. The general equation is thus $y(x) = C_1e^{-2x} + C_2xe^{-2x}$.

5. (10 points) A weight with mass $1/4kg$ is attached to the end of a spring that is stretched $1/4m$ by a force of $9N$. At the start of the experiment, the weight is pulled $0.5m$ to the right, and then let go at an initial velocity of $1ms^{-1}$ to the left. You may ignore the effect of friction.

Write down the initial value problem corresponding to the physical system (i.e. write down the differential equation and the initial values). Do not solve the system.

Solution: Where $x(t)$ is the position function of the weight,

$$\frac{1}{4}x''(t) + 36x(t) = 0, x(0) = 0.5, x'(0) = -1.$$

6. (15 points) Calculate a particular solution to the differential equation

$$y'' + 2y' + y = 2\cos(3x) + 4\sin(3x).$$

Solution: We make the guess $y(x) = A\cos(3x) + B\sin(3x)$, and use the method of undetermined coefficients. We solve for A and B.

We calculate

$$y'(x) = -3A\sin(3x) + 3B\cos(3x), y''(x) = -9A\cos(3x) - 9B\sin(3x).$$

Substituting this into the differential equation, we have that

$$(-9A + 6B + A)\cos(3x) + (-9B - 6A + B)\sin(3x) = 2\cos(3x) + 4\sin(3x).$$

So we have to solve

$$\begin{aligned} -8A + 6B &= 2 \\ -8B - 6A &= 4 \end{aligned}$$

and so from the first equation $B = 1/3 + (4/3)A$. Substituting into the second equation we have $-8/3 - (32/3)A - 6A = 4$, which implies $A = -2/5$, and so $B = -1/5$.

7. (5 points) The solution of an initial value problem involving a forced mass-spring system is given as

$$x(t) = 4\cos(2t) + 5\sin(2t) + e^{-2t}\cos(t) - 3e^{-2t}\sin(t),$$

where x is the position function of the weight on the spring.

Identify the transient and steady periodic solutions.

Solution: The transient solution is

$$e^{-2t}\cos(t) - 3e^{-2t}\sin(t),$$

and the steady periodic solution is

$$4\cos(2t) + 5\sin(2t).$$

8. (10 points) Calculate a general solution for the system

$$x' = y, y' = -x,$$

where x, y are both functions that depend on t, and x', y' both denote derivatives with respect to t.

Solution: We have $y'' = -x' = -y$, and so we need to solve $y'' + y = 0$. Using the substitution $y = e^{rt}$ we get that this is equivalent to $r^2 + 1 = 0$, and so we have roots $r = \pm i$. The general solution for y is $y(t) = C_1 \cos(t) + C_2 \sin(t)$. Also $x(t) = -y'(t) = C_1 \sin(t) - C_2 \cos(t)$.

9. (10 points) Given constants P, Q we have a differential equation

$$y'' + Py' + Qy = 0.$$

However, the polynomial $r^2 + Pr + Q = 0$ has complex roots $r = a \pm ib$ for real numbers a, b, and $i = \sqrt{-1}$. Show that for arbitrary constants C_1, C_2,

$$y = C_1 e^{ax} \cos(bx) + C_2 e^{ax} \sin(bx)$$

is the general solution to the differential equation. You do not need to show that $e^{ax} \cos(bx)$ and $e^{ax} \sin(bx)$ are linearly independent.

Hint: Euler's formula is

$$e^{ik} = \cos(k) + i \sin(k),$$

or alternatively

$$e^{-ik} = \cos(k) - i \sin(k).$$

Solution: We make the substitution $y = e^{rx}$ and the equation changes to

$$e^{rx}(r^2 + Pr + Q) = 0.$$

The roots of the quadratic are $r = a \pm ib$, and so $e^{a \pm ib}$ are two solutions of the differential equation. By the principle of superposition,

$$K_1 e^{(a+ib)x} + K_2 e^{(a-ib)x}$$

is a solution for two arbitrary constants K_1, K_2. We can write this solution as

$$K_1 e^{ax} e^{ibx} + K_2 e^{ax} e^{-ibx},$$

and so using Euler's formula

$$K_1 e^{ax}(\cos(bx) + i \sin(bx)) + K_2 e^{ax}(\cos(bx) - i \sin(bx)),$$

and so this solution can be rewritten as

$$(K_1 + K_2) e^{ax} \cos(bx) + (K_1 i - K_2 i) e^{ax} \sin(bx),$$

and all that is left is to write $C_1 = K_1 + K_2, C_2 = K_1 i - K_2 i$.