MATH 3113-005 Test I

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September 9, 2014 9:00am-10:15am

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. No calculators allowed.

Name: ____

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. (15 points) Solve the initial value problem given by

$$\frac{dy}{dx} = 3\sqrt{x},$$

with initial condition $y(1) = 3$.

Solution: We write

$$\begin{aligned} dy =& 3\sqrt{x}dx \\ y =& 3\int x^{1/2}dx \\ =& 3\frac{x^{3/2}}{3/2} + C \\ y =& 2x^{3/2} + C, \end{aligned}$$

and this last equality serves as our general solution. Our initial condition is $x = 1, y = 3$, so $3 = 2 + C$, and $C = 1$. Our final answer is

$$y = 2x^{3/2} + 1.$$

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2. (15 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = ye^{2x}.$$

Solution: This is a separable equation, so we can write

$$\begin{aligned} \frac{1}{y}dy =& e^{2x}dx \\ \int \frac{1}{y}dy =& \int e^{2x}dx \\ \ln|y| =& \frac{e^{2x}}{2} + C \\ |y| =& e^{\frac{e^{2x}}{2}+C}. \end{aligned}$$

Writing $K = \pm e^C$, we have then

$$y = Ke^{\frac{e^{2x}}{2}}.$$

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3. (10 points) The acceleration dv/dt of a car is proportional to the difference between 200 km/h and the velocity of the car. Write a differential equation that serves as a mathematical model for this situation (You may assume that the car's velocity never exceeds 200 km/h).

Solution: There exists a constant C so

$$\frac{\frac{dv}{dt}}{200 - v} = C.$$

4. (10 points) In the following two problems, the function $x(t)$ denotes a quantity of pollutants inside a vat of liquid. Write down a first-order linear differential equation that models $x(t)$. You **do not** have to solve for the general solution

   a) The vat contains 30 gallons of liquid. Every second, 5 gallons of water with pollutant concentration .05% is poured into the vat, and 5 gallons of liquid is poured out of the vat. Assume that the liquid is perfectly mixed while inside the vat.

   b) The vat contains 50 gallons of liquid. Every second, 1 lb of pollutants enters the tank at 2 gallons per second, and the perfectly mixed solution leaves the tank at a rate of 3 gallons per second.

Solution:

a)

$$x'(t) = \frac{.05}{100}(5) - \frac{x(t)}{30}(5).$$

b) Here, the volume $V(t)$ at time t is $50 + 2t - 3t = 50 - t$. Thus our equation must be

$$x'(t) = \frac{1}{2}(2) - \frac{x(t)}{V(t)}(3),$$

and so, substituting in our value $V(t) = 50 - t$,

$$x'(t) = 1 - \frac{3x(t)}{50 - t}.$$

5. (20 points) Verify that the differential equation

$$(4x - y)dx + (6y - x)dy = 0$$

is exact. Then solve it.

Solution: If this equation is exact, then there exists an $F(x, y)$ for which

$$\frac{dF}{dx} = 4x - y, \frac{dF}{dy} = 6y - x.$$

To verify if this is true, we check that

$$\frac{dF}{dxdy} \stackrel{?}{=} \frac{d(4x - y)}{dy} = -1 = \frac{d(6y - x)}{dx} \stackrel{?}{=} \frac{dF}{dydx}.$$

The equality holds, so the equation is indeed exact.

We then have

$$F = \int 4x - ydx = 2x^2 - xy + g(y),$$

for some function g. Additionally,

$$F = \int 6y - xdy = 3y^2 - xy + h(x),$$

for some function h. Since the two expressions for F must match, it must be true that $g(y) = 3y^2 + C$, and $h(x) = 2x^2 + C$ for some constant C. This leaves us with

$$3y^2 - xy + 2x^2 = C.$$

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6. (20 points) Assume x, y are both positive, and that y is a function of x. Using an appropriate change of variable, rewrite each of these reducible second order differential equations as a first order differential equation in two variables. You **do not** have to solve for the general solution.

   a) $xy'' = y'$.
   b) $yy'' = 6(y')^2$.

Solution:

a) The y is missing, so we can substitute $u = y', u' = y''$ to get

$$xu' = u.$$

b) First, let us rewrite the equation in the $\frac{d}{dx}$ notation:

$$y\frac{d^2y}{dx^2} = 6\left(\frac{dy}{dx}\right)^2.$$

The variable x is missing, so let us make the substitution $u = \frac{dy}{dx}$. We then have

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy}u.$$

Our equation can then be written in the form

$$yu\frac{du}{dy} = 6u^2.$$

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7. (10 points) Consider a differential equation

$$y' + P(x)y = Q(x),$$

with $P(x), Q(x)$ both continuous. Show that

$$y(x) = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right]$$

is a solution to this differential equation for any constant C.

Solution: One way to solve this is to use integrating factors.

$$\begin{aligned}
y' + P(x)y =& Q(x) \\
y'e^{\int P(x)dx} + P(x)e^{\int P(x)dx} =& Q(x)e^{\int P(x)dx} \\
(ye^{\int P(x)dx})' =& Q(x)e^{\int P(x)dx} \\
ye^{\int P(x)dx} =& \int Q(x)e^{\int P(x)dx}dx + C \\
y =& e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right].
\end{aligned}$$

PS. It is also possible to solve it directly by plugging in the given value of $y(x)$ into the differential equation, and then carefully using the product rule and the chain rule.

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