MATH 3113-001 Test I

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September 16, 2015 1:30pm-2:20pm

Answer the questions in the spaces provided on the question sheets. No calculators allowed.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (10 points) Verify that $y(x) = 3\cos(2x)$ is a solution of the differential equation

$$y'' + 4y = 0,$$

by plugging it into the equation.

Solution:

We can calculate that

$$y'(x) = -6\sin(2x), y''(x) = -12\cos(2x).$$

Plugging in y and y'' in our differential equation,we have

$$-12\cos(2x) + 4(3\cos(2x)) = 0,$$

which is true.

2. (20 points) A particle on a line has a constant acceleration of $50ms^{-2}$. Its initial velocity is $6ms^{-1}$, and its initial position is at the 1 meter mark. Calculate the particle's position at time t.

Solution: Let $v(t)$ be the velocity. We have $v'(t) = 50$, and therefore we have a general solution $v(t) = 50t + C$. Since the initial velocity is $v(0) = 6$, we must have $C = v(0) = 6$. Thus the velocity at time t is $v(t) = 50t + 6$.

We can rewrite this as $x'(t) = 50t+6$, where $x(t)$ is the position at time t. Integrating, we obtain a general solution $x(t) = 25t^2 + 6t + K$. We have initial position $x(0) = 1$, and so $K = x(0) = 1$. Thus $x(t) = 25t^2 + 6t + 1$.

3. (20 points) (a) Show that the equation $y' = x^2 + \sin(y)$ has a solution around the initial condition $(1, 0)$, and that this solution is unique.

   (b) Show that the equation $y' = x\sqrt[3]{y}$ has a solution around the initial condition $(1, 0)$. What does the uniqueness test say about whether this solution is unique?

Solution:

(a) $f(x, y) = x^2 + \sin(y)$ is clearly continuous everywhere, and so the equation passes the existence test. We have $\frac{\partial f}{\partial y} = \cos(y)$, which is also clearly continuous everywhere, so the equation passes the uniqueness test.

(b) $f(x, y) = x\sqrt[3]{y}$ is clearly continuous everywhere, so the existence test ensures that there is a solution around $(1, 0)$. However,

$$\frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{\sqrt[3]{y^2}},$$

and this clearly isn't continuous in the region around $(1, 0)$. Thus we cannot guarantee uniqueness.

4. (20 points) A water tank initially contains just 15 gallons of pure water. 5 gallons of waste water enters the tank every second. This waste water has a pollutant concentration of 10%. Assume that the contents of the tank are perfectly mixed. The mixture flows out of the tank at a rate of 5 gallons per second. How much pollutant is inside the tank at time t?

Solution: Let $x(t)$ be the amount of pollutant in the tank at time t. The inflow per second is $5\frac{10}{100} = \frac{1}{2}$. The outflow per second is $\frac{x(t)}{15}5 = \frac{x(t)}{3}$. Thus the equation modeling $x(t)$ is

$$x'(t) = \frac{1}{2} - \frac{x(t)}{3}.$$

We rewrite this as

$$\frac{dx}{dt} + \frac{x}{2} = \frac{1}{3}.$$

The integrating factor is $e^{\int \frac{1}{2} dt} = e^{t/2}$. Multiplying both sides by this integrating factor we have

$$\frac{dx}{dt}e^{t/2} + \frac{xe^{t/2}}{2} = \frac{e^{t/2}}{3}$$

We simplify the left side of the equation to

$$\frac{d(xe^{t/2})}{dt} = \frac{e^{t/2}}{3}.$$

We integrate both sides to obtain

$$xe^{t/2} = \int \frac{e^{t/2}}{3} dt = \frac{2}{3}e^{t/2} + C$$

Dividing both sides by $e^{t/2}$ we obtain the general solution

$$x(t) = 2/3 + C/e^{t/2}.$$

We know that when $t = 0$, there is no pollutant in the tank, so we have the initial condition is $x(0) = 0$. Thus

$$0 = x(0) = \frac{2}{3} + \frac{C}{e^{0/2}} = \frac{2}{3} + C.$$

We conclude that $C = -2/3$, and so the amount of pollutant in the tank at time t is $x(t) = 2/3 - 2e^{-t/2}/3$ gallons.

5. (20 points) Find the general solution for

$$y'' = \frac{y'}{x+1}.$$

For convenience, you may assume that x and y' are both positive.

Solution: We make the substitution $u = \frac{dy}{dx}$, and so $\frac{du}{dx} = \frac{d^y}{dx^2}$. Thus we may rewrite our differential equation as

$$\frac{du}{dx} = \frac{u}{x+1}.$$

This is a separable equation, so we rewrite it as

$$\int \frac{1}{u} du = \frac{1}{x+1} dx.$$

Since y' is positive, u must be postive. We also know that x is positive, so we can integrate both sides to get

$$\ln(u) = \ln(x+1) + C.$$

Exponentiating both sides we obtain $u = (x+1)e^C$. Let us re-label the constant $K = e^C$ to obtain the general solution

$$u = (x+1)K$$

We plug $u = \frac{dy}{dx}$ back in to obtain the equation

$$\frac{dy}{dx} = (x+1)K.$$

Integrating both sides we have the general solution

$$y = (x^2 + x + D)K,$$

for another arbitrary constant D.

6. (10 points) Nissa wants to calculate the general solution of the equation

$$y' = \sin(x) + \cos(x).$$

She integrates both sides of this equation to obtain

$$y = -\cos(x) + \sin(x).$$

Did Nissa obtain the correct general solution? If you think it is correct, verify it is the general solution by plugging it in the differential equation. If you do not think it is correct, explain why this isn't the general solution.

Solution: This is not the general solution. A general solution is a collection of all possible solutions of a differential equation. Nissa only found a single solution, because she forgot to add the integration constant $+C$.