Midterm Exam I
Math 212 Summer General Session 2011

Instructions: This is a **90**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

1. (20 points total) Calculate $\frac{\mathbf{u}}{||\mathbf{u}||}$, $\frac{\mathbf{v}}{||\mathbf{v}||}$, $\mathbf{u}\cdot\mathbf{v}, \mathbf{u}\times\mathbf{v}$, and $||\mathbf{u}-\mathbf{v}||$ where

$$\mathbf{u} = \mathbf{i} + 3\mathbf{j}, \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}.$$

Answers

$$\begin{aligned}
\frac{\mathbf{u}}{||\mathbf{u}||} =& (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \\
\frac{\mathbf{v}}{||\mathbf{v}||} =& (\frac{-3}{\sqrt{70}}, \frac{5}{\sqrt{70}}, \frac{6}{\sqrt{70}}) \\
\mathbf{u}\cdot\mathbf{v} =& -6+5+6=5 \\
\mathbf{u}\times\mathbf{v} =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -3 & 5 & 6 \end{vmatrix} = (6-5)\mathbf{i} - (12+3)\mathbf{j} + (10+3)\mathbf{k} \\
=& \mathbf{i} - 15\mathbf{j} + 13\mathbf{k} \\
||\mathbf{u}-\mathbf{v}|| =& \sqrt{(2+3)^2 + (1-5)^2 + (1-6)^2} = \sqrt{66}.
\end{aligned}$$

2. (20 points total) Consider a triangle with vertices $P = (3,1,2), Q = (3,-1,5), R = (5,-1,2)$.

   (a) Find the area of the triangle.(10 points)

   (b) Calculate $\cos(\theta)$, where θ is the angle of the triangle at the vertex Q. (10 points)

Answers We have $\vec{QP} = (0,2,-3)$ and $\vec{QR} = (2,0,-3)$.

(a) Then

$$
\begin{aligned}
\text{Area}(\Delta PQR) =& \frac{1}{2}||\vec{QP} \times \vec{QR}|| \\
=& \frac{1}{2}\left|\left|\left(\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -3 \\ 2 & 0 & -3 \end{vmatrix}\right)\right|\right| \\
=& \frac{1}{2}||-6\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}|| \\
=& \sqrt{22}.
\end{aligned}
$$

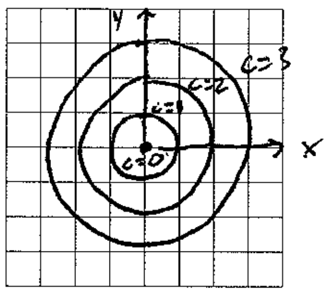
(b)

$$
\begin{aligned}
\cos\theta =& \frac{\vec{QP} \cdot \vec{QR}}{(||\vec{QP}||)(||\vec{QR}||)} \\
=& \frac{0+0+9}{\sqrt{13}\sqrt{13}} \\
=& \frac{9}{13}
\end{aligned}
$$

3. (15 points total) Consider the surface defined as the graph of the function

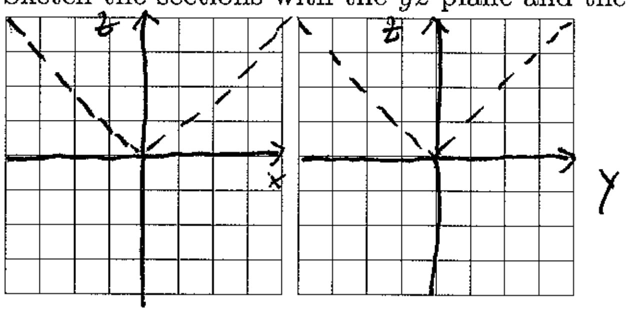
$$f(x, y) = \sqrt{x^2 + y^2}.$$

(a) Sketch the level curves of levels $c = -1, 0, 1, 2, 3$. (5 points)



$c=-1$	no level curve	
$c=0$	$x^2+y^2=0$	point at $(0,0)$
$c=1$	$x^2+y^2=1$	circle radius 1
$c=2$	$x^2+y^2=2^2$	circle radius 2
$c=3$	$x^2+y^2=3^2$	circle radius 3.

(b) Sketch the sections with the yz-plane and the xz-plane. (5 points)



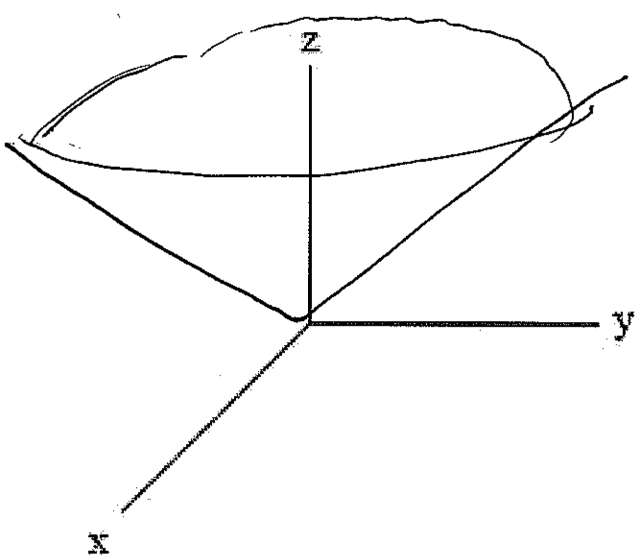
$z = f(x,0) = \sqrt{x^2} = |x|$

$z = f(0,y) = \sqrt{y^2} = |y|$

section with xz-plane ($y=0$)

section with yz-plane ($x=0$)

(c) Sketch the surface. (5 points)



4. (15 points total) Find the following limits, or show that they don't exist.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2y+1}.$$

(5 points)

(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

(5 points)

(c)

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1-xy}{x^2y^2}.$$

(5 points)

Answers

(a)

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2y+1} = \frac{\cos(0)}{0+1} = 1$$

(b) If we consider the path $(0,t), t \to 0$ the limit would be equal to

$$\lim_{t\to 0} \frac{0t}{0^2+t^2} = 0.$$

However, if we consider the path $(t,t), t \to 0$ the limit would be equal to

$$\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \frac{1}{2}.$$

We conclude that the limit does not exist.

(c) We have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1-xy}{x^2y^2} &= \lim_{(x,y)\to(0,0)} \frac{\left(1+xy+\frac{(xy)^2}{2!}+\frac{(xy)^3}{3!}+\dots\right)-1-xy}{x^2y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{\frac{(xy)^2}{2!}+\frac{(xy)^3}{3!}+\dots}{x^2y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{1}{2!}+\frac{xy}{3!}+\dots \\
&= \frac{1}{2}
\end{aligned}$$

5. (10 points total) A submarine is located at the point $(3, 4)$. That is, it is located 3 miles east of the origin and 4 miles north of the origin. It is submerged at a depth of 12 miles. A battleship is located at the point $(-5, 6)$ on the surface of the ocean. The submarine fires a torpedo that heads towards the surface at a rate of 3 miles per second, west at a rate of 2 miles per second, and north at a rate of 0.5 miles per second.

   (a) How long does it take for the torpedo to reach the surface of the ocean? (5 points)

   (b) Does the torpedo hit the battleship? (10 points)

   (c) What if the battleship were an iceberg? (0 points)

Answers

(a) The torpedo starts at a depth of 12 miles and is heading toward the surface at a rate of 3 miles per second. Thus it takes the torpedo 4 seconds to reach the surface.

(b) The location of the torpedo at time t is $(3, 4, -12) + t(-2, 0.5, 3)$. At time $t = 4$, the torpedo is at location $(-5, 6, 0)$ and so yes, it does hit the battleship.

(c) If ifs and buts were candy and nuts, we'd all have a merry Christmas.

6. (15 points total) Define $h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and

$$\begin{aligned} u(x,y,z) =& x + y + z, \\ v(x,y,z) =& yze^x, \\ w(x,y,z) =& 3y. \end{aligned}$$

If we are given that, at the point $(0,-1,3)$

$$\frac{\partial f}{\partial u} = 2, \frac{\partial f}{\partial v} = -3, \frac{\partial f}{\partial w} = 1,$$

calculate $\nabla h(0,-1,3)$. (10 points)

Answers Clearly $(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. Thus then

$$\begin{aligned} \nabla h(0,-1,3) =& \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) (\text{ at } x = 0, y = -1, z = 3) \\ =& \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}\right) \begin{pmatrix} 1 & 1 & 1 \\ yze^x & ze^x & ye^x \\ 0 & 3 & 0 \end{pmatrix} (\text{ at } x = 0, y = -1, z = 3) \\ =& (2,-3,1) \begin{pmatrix} 1 & 1 & 1 \\ -3 & 3 & -1 \\ 0 & 3 & 0 \end{pmatrix} \\ =& (11,-4,5). \end{aligned}$$