Practice Midterm Exam II
Math 101 Summer General Session 2010

Instructions: This is a **105**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
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8	
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10	
11	
Total	

Some useful identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\pi \text{ radians} = 180^\circ$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\log(ab) = \log a + \log b$$

$$\log(a^b) = b\log a$$

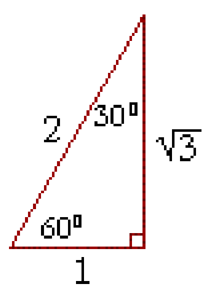
$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

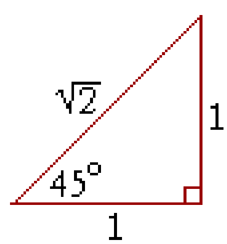
$$\ln(x) = \log_e(x)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$





You may use the above triangles to recall certain values of sin, cos, and tan.

1. (10 points)

   (a) Calculate

$$\sum_{i=1}^{5}(i+1).$$

   (b) Express

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$$

   in Σ-notation.

Answers.

(a)

$$\begin{aligned}\sum_{i=1}^{5}(i+1) =& 2+3+4+5+6\\ =& 20.\end{aligned}$$

(b)

$$\sum_{i=1}^{6}\frac{1}{i}.$$

2. (10 points) Use the technique of linear approximation to estimate the following quantities:

   (a) $\sqrt{82}$.

   (b) the change in the area of a square, if its edge length decreases from 10cm to 9.8cm.

Answers. The linear approximation formula is $f(x) \approx f(a) + f'(a)(x-a)$.

(a) We take $f(x) = \sqrt{x}$, $a = 81$ and $x = 82$. This means that $f'(x) = 1/2\sqrt{x}$. The approximation formula tells us that

$$\begin{aligned}\sqrt{82} \approx& \sqrt{81} + \frac{1}{2\sqrt{81}}(82-81) \\ =& 9 + 1/18.\end{aligned}$$

(b) We take $f(x) = x^2$, $x = 9.8$, $a = 10$ and $f'(x) = 2x$. The approximation formula gives us

$$\begin{aligned}f(9.8) - f(10) \approx& f'(9.8)(9.8-10) \\ =& 19.6(-0.2) = -3.92\end{aligned}$$

Notes. When doing an approximation of value problem, a is the 'known' value (in this case $a = 81$) and x is the unknown value (in this case $x = 82$). When doing an approximation of change problem, a is the old value (in this case $a = 10$) and x is the new value (in this case $x = 9.8$).

3. (5 points) Joe is driving along a highway with speed limit 70 miles per hour. He passes through two toll booths that are thirty miles apart. He reaches the first toll booth at 2:10pm, and the second toll booth at 2:35pm. Prove that at some point, Joe was speeding.

Answers. We know that at 2:10 pm Joe is at the position 0, and at 2:35 pm Joe is at position 30. Thus, if we consider the position function $x(t)$ as a graph on the tx-plane, the position function touches the points (2:10, 0) and (2:35, 30). Thus by the mean value theorem, for some t_0 between 2:10 and 2:35,

$$\begin{aligned} x'(t_0) =& \frac{30-0}{2:35-2:10} \\ =& \frac{30}{25} \text{ miles/minute} \\ =& \frac{30}{\frac{25}{60}} \text{miles/hour} \\ =& 72 \text{ miles/hour} \end{aligned}$$

4. (15 points) Find $f'(x)$, $f''(x)$, and $f'''(x)$ when

(a) $f(x) = \sin(3x)$.

(b) $f(x) = x^3 + x^2$.

(c) $f(x) = e^{-x}$.

Answers.

(a) $f'(x) = 3\cos(3x), f''(x) = -9\sin(3x), f'''(x) = -27\cos(3x)$.

(b) $f'(x) = 3x^2 + 2x,\ f''(x) = 6x + 2,\ f'''(x) = 6$.

(c) $f'(x) = -e^{-x}, f''(x) = e^{-x}, f'''(x) = -e^{-x}$.

Notes. Part (a) is solved using the chain rule for $u = 3x$. Part (b) is solved using the power rule. Part (c) is solved using the chain rule for $u = -x$.

5. (15 points) Make a detailed sketch of the graph $y = \frac{\ln x}{x}$ on $(0, \infty)$. Make sure that you clearly note

   (a) all inflection points and all maxima/minima.

   (b) where the graph is increasing,decreasing, concave up or concave down.

   (c) the horizontal and vertical asymptotes. There is one of each. Write down the equations for those asymptotes.

Answers. If $f(x) = \frac{\ln(x)}{x}$ we have $f'(x) = \frac{\frac{1}{x}x - \ln(x)\cdot 1}{x^2} = \frac{1-\ln(x)}{x^2}$. Setting $f'(x) = 0$ gets us $\ln(x) = 1$, and this only happens when $x = e$. We can calculate that $f(e) = 1/e$. Thus we possibly have one extrema at the point $(e, 1/e)$. (Recall that e is approximately 2.71). We pick one number between 0 and e and one number between e and ∞). Let our choices be 1 and e^2 respectively. Since $f'(1) = 1 > 0$ and $f'(e^2) = -1/e^4 < 0$, we know that $(e, 1/e)$ is a maxima, and since it is the only critical point it must be an absolute maxima. We also know that f is increasing in the interval between $x = 0$ and $x = e$ and decreasing in the interval between $x = e$ and $x = \infty$.

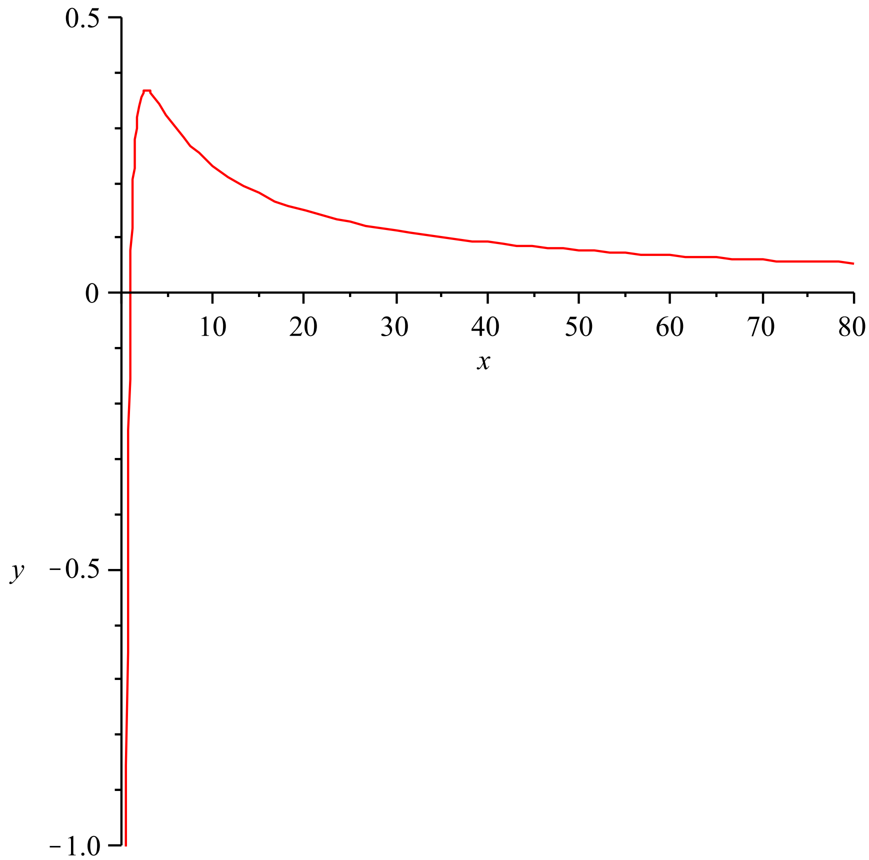
We calculate $f''(x) = \frac{-\frac{1}{x}x^2 - 2x(1-\ln(x))}{x^4} = \frac{-3+2\ln(x)}{x^3}$. So $f''(x) = 0$ only when $-3 + 2\ln(x) = 0$, and this happens only when $x = e^{\frac{3}{2}}$. Since $f(e^{\frac{3}{2}}) = \frac{3}{2e^{\frac{3}{2}}}$, we possibly have one inflection point in $(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}})$. We pick one number between 0 and $e^{\frac{3}{2}}$ and another number between $e^{\frac{3}{2}}$ and ∞. Let our choices be 1 and e^2 respectively. We have $f''(1) = -3 < 0$, and $f''(e^2) = 1/e^6 > 0$, and thus $(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}})$ is indeed an inflection point, and f is concave down in between $x = 0$ and $x = e^{\frac{3}{2}}$ and concave up between $x = e^{\frac{3}{2}}$ and $x = \infty$.

We need to check asymptotes on the boundaries of the domain (that is, $x = 0$ and $x = \infty$). We can check that $\lim_{x\to 0} \ln(x)/x = -\infty$ (Since $-\infty/0 \approx -\infty$) and that

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

by l'Hôpital's rule. Thus there are asymptotes at $x = 0$ and $y = 0$.

Notes. Note that $f'(x)$ and $f''(x)$ are both undefined at $x = 0$- however, we don't consider $x = 0$ when calculating inflection points or critical points because $x = 0$ is not in the domain.



6. (10 points)Use l'Hôpital's rule to find the following limits:

(a)

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

(b)

$$\lim_{x\to 0^+} x^x.$$

Answers.

(a)

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{e^x} &= \lim_{x\to\infty} \frac{2x}{e^x} \text{ (l'Hôpital with } \frac{\infty}{\infty}\text{)}\\ &= \lim_{x\to\infty} \frac{2}{e^x} \text{ (l'Hôpital with } \frac{\infty}{\infty}\text{)}\\ &\approx \frac{2}{\infty} = 0\end{aligned}$$

(b) Let $y = x^x$

$$\begin{aligned}\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} x\ln(x)\\ &= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \text{ (l'Hôpital with } \frac{\infty}{\infty}\text{)}\\ &= \lim_{x\to 0^+} \frac{1}{-\frac{1}{x}}\\ &= \lim_{x\to 0^+} -x\\ &=0\end{aligned}$$

And since $\lim \ln(x^x) = 0$, we know that $\lim x^x = \lim e^{\ln(x^x)} = e^{\lim \ln(x^x)} = e^0 = 1$.

7. (10 points) Find the following antiderivatives:

(a)

$$\int x(x+1)dx.$$

(b)

$$\int \sin(x) - 2\cos(x)dx.$$

Answers.

(a) $\int x(x+1)dx = \int(x^2+x)dx = \frac{x^3}{3} + \frac{x^2}{2} + C.$

(b) $\int \sin(x) - 2\cos(x) = \int \sin(x) - 2\int \cos(x) = -\cos(x) - 2\sin(x) + C.$

Notes Forgetting the "$+C$" will lose you one point for each of these problems. Do not forget the "$+C$"!

8. (10 points) Calculate the following definite integrals:

(a)

$$\int_0^1 (2x+1)^5 dx.$$

(b)

$$\int_{-1}^2 \frac{x+1}{(x^2+2x-1)^2}.dx$$

Answers.

(a) For the first method, we make the substitution $u = 2x+1$, so $\dfrac{du}{dx} = 2$ and $dx = \frac{1}{2}du$. Thus when $x = 1, u = 3$ and when $x = 0$, $u = 1$- this gives us the bounds.

$$\begin{aligned}\int_0^1 (2x+1)^5 dx &= \int_1^3 u^5 \frac{1}{2} du \\ &= \left[\frac{u^6}{12}\right]_1^3 \\ &= \left[\frac{3^6}{12}\right] - \left[\frac{1}{12}\right] = \frac{182}{3}.\end{aligned}$$

Alternatively, you can use the second method,

$$\begin{aligned}\int_0^1 (2x+1)^5 dx &= \int_{x=0}^{x=1} u^5 \frac{1}{2} du \\ &= \left[\frac{u^6}{12}\right]_{x=0}^{x=1} \\ &= \left[\frac{(2x+1)^6}{12}\right]_{x=0}^{x=1} \\ &= \left[\frac{3^6}{12}\right] - \left[\frac{1}{12}\right] = \frac{182}{3}.\end{aligned}$$

(b) For the first method, we make the substitution $u = x^2 + 2x - 1$, so $\dfrac{du}{dx} = 2x + 2 = 2(x+1)$ and $(x+1)dx = \frac{1}{2}du$. Thus when $x = 2, u = 7$ and when $x = -1$, $u = -2$- this gives us the bounds.

$$\begin{aligned}
\int_{-1}^{2} \frac{x+1}{(x^2+2x-1)^2}dx &= \int_{-2}^{7} \frac{\frac{1}{2}}{u^2}du \\
&= \int_{-2}^{7} \frac{u^{-2}}{2}du \\
&= \left[\frac{u^{-1}}{-2}\right]_{-2}^{7} \\
&= \left[\frac{1}{-2u}\right]_{-2}^{7} \\
&= \left[\frac{1}{-14}\right] - \left[\frac{1}{4}\right] = -\frac{9}{28}.
\end{aligned}$$

Alternatively, you can use the second method,

$$\begin{aligned}
\int_{-1}^{2} \frac{x+1}{(x^2+2x-1)^2}dx &= \int_{x=-1}^{x=2} \frac{\frac{1}{2}}{u^2}du \\
&= \int_{x=-1}^{x=2} \frac{u^{-2}}{2}du \\
&= \left[\frac{u^{-1}}{-2}\right]_{x=-1}^{x=2} \\
&= \left[\frac{1}{-2u}\right]_{x=-1}^{x=2} \\
&= \left[\frac{1}{-2(x^2+2x-1)}\right]_{x=-1}^{x=2} \\
&= \left[\frac{1}{-14}\right] - \left[\frac{1}{4}\right] = -\frac{9}{28}.
\end{aligned}$$

9. (5 points) A penny dropped from the top of a building lands on the pavement three seconds later. Acceleration due to gravity is 32 feet per second per second. How tall is the building?

Answers. Let $x(t)$ be the position function of the penny, and let the top of the building be position 0, so $x(0) = 0$. Since the penny is dropped (and not, say, thrown) , we must have $x'(0) = 0$, since the velocity of a dropped penny is initially zero. But then $x''(t) = a(t) = -32$, and so $x'(t) = \int x''(t)dt = -32t + C$ for some constant C. Since $x'(0) = 0$, $C = 0$. Thus $x'(t) = -32t$.

We also know that $x(t) = \int x'(t)dt = -16t^2 + C$. But since $x(0) = 0$, C again is 0 and so $x(t) = -16t^2$. $x(3) = -144$, and thus the penny is 144 feet below the top of the building after three seconds. Hence the building must be 144 feet tall.

10. (5 points) Find the area of the planar region bounded below by the line $y = 16$ and bounded above by the parabola $y = 25 - x^2$.

Answer. We first need to figure out where the two curves meet: we have to solve for $25 - x^2 = 16$, and this gets us $x^2 = 9$ or $x = \pm 3$. This gives us the bounds of our integral. The area of the planar region must thus be

$$\begin{aligned}\int_{-3}^{3} [(25 - x^2) - 16] dx &= \int_{-3}^{3} (9 - x^2) dx \\ &= \left[9x - \frac{x^3}{3} \right]_{-3}^{3} \\ &= [18] - [-18] = 36\end{aligned}$$

Notes If you are wondering what this region looks like, it is similar to Figure 5.8.15 on page 390 of your textbook.

11. (5 points) Estimate $\int_0^\pi f(x)dx$ using Simpson's Approximation with $n = 4$, where $f(x) = \frac{\sin(x)}{x}$ when x nonzero, and $f(0) = 1$.

Answer. We have $\Delta x = \pi/4$, $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$ and $x_4 = \pi$. This yields us

$$\begin{aligned}
f(x_0) =& f(0) = 1, \\
f(x_1) =& f(\pi/4) = \frac{\sin(\pi/4)}{\pi/4} = \frac{\frac{1}{\sqrt{2}}}{\pi/4} = \frac{2\sqrt{2}}{\pi}, \\
f(x_2) =& f(\pi/2) = \frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}, \\
f(x_3) =& f(3\pi/4) = \frac{\sin(3\pi/4)}{3\pi/4} = \frac{\frac{1}{\sqrt{2}}}{3\pi/4} = \frac{2\sqrt{2}}{3\pi}, \\
f(x_4) =& f(\pi) = \frac{\sin(\pi)}{\pi} = 0.
\end{aligned}$$

We can then find the Simpsons' Approximation S_4:

$$\begin{aligned}
S_4 =& \frac{(\pi/4)}{3}\left(1(1) + 4\left(\frac{2\sqrt{2}}{\pi}\right) + 2\left(\frac{2}{\pi}\right) + 4\left(\frac{2\sqrt{2}}{3\pi}\right) + 1(0)\right) \\
=& \frac{1}{12}\left(\pi + \left(8\sqrt{2}\right) + 4 + \left(\frac{8\sqrt{2}}{3}\right)\right) \\
=& \frac{1}{12}\left(\pi + 4 + \left(\frac{32\sqrt{2}}{3}\right)\right)
\end{aligned}$$

Notes. You don't have to simplify the answer more than this.