Practice Midterm Exam I
Math 101 Summer General Session 2010

Instructions: This is a **105**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Some useful identities

$\sin^2\theta + \cos^2\theta = 1$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\pi \text{ radians} = 180^\circ$$

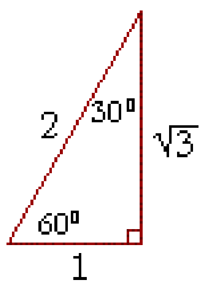
$$a^2 - b^2 = (a+b)(a-b)$$

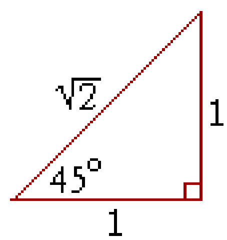
$$\log(ab) = \log a + \log b$$

$$\log(a^b) = b\log a$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\ln(x) = \log_e(x)$$





You may use the above triangles to recall certain values of sin, cos, and tan.

1. (20 points) Consider the polynomial function $f(x) = x^3 + 3x^2 - 2x - 6$.

   (a) Calculate $f(2)$ and $f\left(-\frac{1}{3}\right)$.

   (b) Find the derivative $f'(x)$.

   (c) Find the equation of the tangent line of $y = f(x)$ when $x = 1$.

   (d) Show that $f(x) = 0$ for some x between $x = 1$ and $x = 2$.

Answer

(a) $f(2) = 8 + 12 - 4 - 6 = 10$, $f(-1/3) = -1/27 + 1/3 + 2/3 - 6 = -136/27$.

(b)

$$\begin{aligned} f'(x) =& 3x^2 + 3(2)x - 2 \\ =& 3x^2 + 6x - 2. \end{aligned}$$

(c) The derivative $f'(1) = 7$, and so the tangent line has slope 7. Also $f(1) = -4$, and so the point on the curve has coordinates $(1, -4)$. Hence we are looking for the equation of a line with slope 7 through the point $(1, -4)$.

The general equation of a line is $y = mx + c$. Substituting $m = 7$ we get $y = 7x + c$. We now need to solve for c. We know that the line goes through $(1, -4)$, so we have the equation $-4 = 7(1) + c$, and so $c = -11$. The equation of our tangent line must then be $y = 7x - 11$.

(d) We can calculate that $f(1) = -4$ and $f(2) = 10$. Since one of these values is positive and the other is negative, by the Intermediate Value Theorem we must have a root between $x = 1$ and $x = 2$.

2. (10 points) Calculate the following limits:

(a)

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos x}{x^2} \right).$$

(b)

$$\lim_{x \to 0} x^2 \sin \left(\frac{\sqrt[3]{x+3}}{x} \right).$$

Answer

(a) We use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\cos(0) = 1$.

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos x}{x^2} \right) &= \lim_{x \to 0} \left(\frac{1 - \cos x}{x^2} \right) \\
&= \lim_{x \to 0} \left(\frac{(1 - \cos x)}{x^2} \frac{(1 + \cos x)}{(1 + \cos x)} \right) \\
&= \lim_{x \to 0} \left(\frac{1 - \cos^2 x}{x^2(1 + \cos x)} \right) \\
&= \lim_{x \to 0} \left(\frac{\sin^2 x}{x^2(1 + \cos x)} \right) \\
&= \lim_{x \to 0} \left(\frac{\sin x}{x} \frac{\sin x}{x} \frac{1}{1 + \cos x} \right) \\
&= (1)(1) \left(\frac{1}{1+1} \right) = \frac{1}{2}.
\end{aligned}$$

(b) We use the Squeeze Theorem. Since

$$-1 \le \sin \left(\frac{\sqrt[3]{x+3}}{x} \right) \le 1,$$

We must have

$$-x^2 \le x^2 \sin \left(\frac{\sqrt[3]{x+3}}{x} \right) \le x^2.$$

But then $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} -x^2 = 0$. Thus the middle limit,

$$\lim_{x \to 0} x^2 \sin \left(\frac{\sqrt[3]{x+3}}{x} \right) = 0.$$

3. (5 points) What is the largest possible domain of the function

$$f(x) = \frac{\sqrt{x+1}}{|x|}?$$

Answer Since the numerator contains $\sqrt{x+1}$, we need $x + 1 \geq 0$. Since $|x|$ is in the denominator, we need $x \neq 0$. But $x + 1 \geq 0$ implies $x \geq -1$, and so our largest possible domain is $x \geq -1$ except for $x = 0$.

4. (15 points) Find the derivative $\frac{dy}{dx}$ when

(a) $y = x^2 + 1$.

(b) $y = x^2 \sin(x)$.

(c) $y = \cos(2x + 3)$.

Answer

(a) $\frac{dy}{dx} = 2x$, by the power rule.

(b) By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dx^2}{dx}\sin x + x^2 \frac{d\sin x}{dx} \\ &= 2x\sin x + x^2 \cos x\end{aligned}$$

(c) We set $u = 2x + 3$ (and so $y = \cos u$). We can then use the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\sin u \cdot 2 \\ &= -2\sin(2x + 3).\end{aligned}$$

5. (15 points) Find the derivative $f'(x)$ when

(a) $f(x) = \ln(x \cos x)$.

(b) $f(x) = e^{\sqrt{x}}$.

(c) $f(x) = \frac{e^x}{\sqrt[3]{x}}$.

Answers

(a) It is possible to calculate this using the chain rule and then the product rule, but the best way to handle this is to first invoke the properties of the logarithm. Also, recall that the derivative of $\ln(f(x))$ is $f'(x)/f(x)$, for any differentiable function f.

$$\begin{aligned} f(x) &= \ln(x) + \ln(\cos(x)) \\ f'(x) &= \frac{1}{x} + \frac{-\sin x}{\cos x} \end{aligned}$$

(b)

$$\begin{aligned} f(x) &= e^{x^{\frac{1}{2}}} \\ f'(x) &= e^{x^{\frac{1}{2}}} \left(\frac{1}{2} x^{-\frac{1}{2}} \right) \end{aligned}$$

(c) We invoke the quotient rule

$$\begin{aligned} f(x) &= \frac{e^x}{x^{1/3}} \\ f'(x) &= \frac{e^x x^{1/3} - e^x (1/3) x^{-2/3}}{(x^{1/3})^2} \\ &= \frac{e^x \left(x^{1/3} - \left(\frac{x^{-2/3}}{3} \right) \right)}{x^{2/3}} \end{aligned}$$

6. (5 points) Find the absolute maximum and absolute minimum values for $f(x) = x^2+6x+11$ in the interval $[-4, -1]$.

First, we find the derivative

$$f'(x) = 2x + 6.$$

We need to find the critical points, and so we solve for $f'(x) = 0$. This gives us

$$2x + 6 = 0$$

$$x = -3$$

which is a critical value in the interval $[-4, -1]$. Our candidates are thus $x = -4, x = -1$, and $x = -3$. We have

$$f(-4) = 16 - 24 + 11 = 3$$

$$f(-1) = 1 - 6 + 11 = 6$$

$$f(-3) = 9 - 18 + 11 = 2$$

Our maximum value is 6, and our minimum value is 2.

7. (10 points) A rectangle has perimeter of 100 meters. What is its largest possible area?

Answer Let the width of our rectangle be x, and let the height of our rectangle be y. Then our perimeter is $2x + 2y = 100$. Also, clearly the width x of our rectangle cannot exceed 50 meters, since our perimeter is 100 meters.

Thus we want to maximize the area $f(x) = xy$ with $0 \leq x \leq 50$. The equation $2x+2y = 100$ implies that $y = 50 - x$, and so we can rewrite $f(x)$ as $f(x) = x(50 - x) = 50x - x^2$. We take the derivative $f'(x) = 50 - 2x$. We look for the critical values, and so we need to solve for $f'(x) = 0$. This gets us $50 - 2x = 0$, or $x = 25$.

Our candidates for the maximum value are at $x = 0, x = 50$, and $x = 25$. But recalling that $f(x) = x(50 - x)$, we have

$$\begin{aligned} f(0) =& 0(50-0) = 0 \\ f(50) =& 50(50-50) = 0 \\ f(25) =& 25(25) = 625 \end{aligned}$$

Thus the maximum area of the rectangle is 625 square meters.

8. (5 points) The sides of a square are all increasing at a rate of 0.5 centimeters per second at the moment that they have length 10cm. How quickly is the area of the square increasing at that time?

Answer If the square has side length x, it has area $A = x^2$. We differentiate that equation in terms of t

$$\begin{aligned} \frac{d}{dt}(A) &= \frac{d}{dt}\left(x^2\right) \\ \frac{dA}{dt} &= \frac{dx^2}{dx}\frac{dx}{dt} \\ \frac{dA}{dt} &= 2x\frac{dx}{dt} \end{aligned}$$

But we know that when $x = 10$, $\frac{dx}{dt} = 0.5$. and so the above equation yields

$$\frac{dA}{dt} = 2(10)(0.5) = 10.$$

And so our area is increasing at a rate of 10 square centimers per second.