Midterm Exam II
Math 101 Summer General Session 2010

Instructions: This is a **105**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Some useful identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\pi \text{ radians} = 180°$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\log(ab) = \log a + \log b$$

$$\log(a^b) = b\log a$$

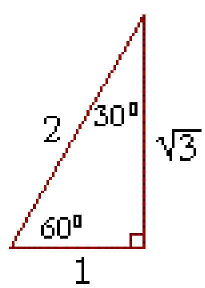
$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

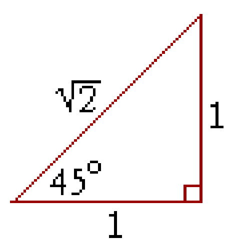
$$\ln(x) = \log_e(x)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$





You may use the above triangles to recall certain values of sin, cos, and tan.

1. (10 points)

   (a) Calculate

$$\sum_{i=1}^{5}(i^2+i).$$

   (b) Calculate

$$\sum_{i=1}^{100}(i-1).$$

Answers.

(a)

$$\sum_{i=1}^{5}(i^2+i) = 2+6+12+20+30 = 70.$$

(b)

$$\begin{aligned}\sum_{i=1}^{100}(i-1) &= \sum_{i=1}^{100} i - \sum_{i=1}^{100} 1 \\ &= \frac{(100)(101)}{2} - 100 = 4950.\end{aligned}$$

2. (10 points)

(a) Find $f'(x)$ when $f(x) = \int_3^x \cos(\sin(\cos(t)))dt$.

(b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when y is defined implicitly as a function of x by the equation

$$x^2 + y^2 = 4.$$

(a) By the Fundamental Theorem of Calculus, $f'(x) = \cos(\sin(\cos(x)))$.

(b)

$$\begin{aligned}
x^2 + y^2 &= 4 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}4 \\
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-2x}{2y} \\
&= \frac{-x}{y}. \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{-x}{y}\right) \\
&= \frac{\frac{d(-x)}{dx}y - (-x)\left(\frac{dy}{dx}\right)}{y^2} \\
&= \frac{-y + x\frac{dy}{dx}}{y^2} \\
&= \frac{-y + x\left(\frac{-x}{y}\right)}{y^2}.
\end{aligned}$$

3. (10 points) Calculate $\int_0^2 x^2 dx$ by

   (a) using the formula
$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b.$$

   (b) using the formula
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x,$$
where Δx is $(b-a)/n$, the length of the interval partitioned into n parts, and $x_i = a + i\Delta x$.

Answers.

(a) $\int_0^2 x^2 dx = [x^3/3]_0^2 = 8/3$.

(b) We have $a = 0, b = 2, \Delta x = 2/n, x_i = 2i/n$. So

$$\begin{aligned}
\int_0^2 x^2 dx &= \lim_{n\to\infty}\sum_{i=0}^{n} f\left(\frac{2i}{n}\right)\Delta x. \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{2i}{n}\right)^2 \frac{2}{n}. \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty}\frac{8}{n^3}\sum_{i=0}^{n} i^2 \\
&= \lim_{n\to\infty}\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}. \\
&= \lim_{n\to\infty}\frac{16n^3+24n^2+8n}{6n^3} \\
&= \lim_{n\to\infty}\frac{16n^3+24n^2+8n}{6n^3}\left(\frac{1/n^3}{1/n^3}\right) \\
&= \lim_{n\to\infty}\frac{16+24/n+8/n^2}{6} \\
&= \frac{16}{6} = \frac{8}{3}
\end{aligned}$$

4. (15 points) Find $f'(x)$, $f''(x)$, and $f'''(x)$ when

(a) $f(x) = -x$.

(b) $f(x) = \ln(x)$.

(c) $f(x) = \sin(\pi x)$.

(a) $f'(x) = -1, f''(x) = 0, f'''(x) = 0$.

(b) $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}$.

(c) $f'(x) = \pi\cos(\pi x), f''(x) = -\pi^2\sin(\pi x), f'''(x) = -\pi^3\cos(\pi x)$.

5. (10 points) Make a detailed sketch of the graph $y = 6x^4 - 8x^3 + 1$. Make sure that you clearly note

(a) all inflection points and all maxima/minima.

(b) where the graph is increasing,decreasing, concave up or concave down.

Answers. The derivative is $f'(x) = 24x^3 - 24x^2 = 24x^2(x-1)$. So $f'(x) = 0$ implies that $x = 0$ or $x = 1$. We can check that $f(-1) < 0, f(1/2) < 0$, and $f(2) > 0$. Thus the function is decreasing on $(-\infty, 0)$ and $(0, 1)$, and increasing on $(1, \infty)$. This implies that we have a minima at $x = 1$. Since $f(1) = -1$, we can identify this minima point as $(1, -1)$.

The second derivative is $f''(x) = 72x^2 - 48x = 24x(3x - 2)$. Thus if $f''(x) = 0, x = 0$ or $x = 2/3$. We can check that $f(-1) > 0$, $f(1/2) < 0$, and $f(2) > 0$. Thus f is concave up on $(-\infty, 0)$ and $(2/3, \infty)$, and concave down on $(0, 2/3)$. We have two inflection points at $(0, 1)$ and $(2/3, -15/81)$. This enough information to sketch the graph.

6. (15 points)Use l'Hôpital's rule to find the following limits:

(a)

$$\lim_{x\to\infty} \frac{x+1}{xe^x}.$$

(b)

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x.$$

(c)

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right).$$

Answers.

(a)

$$\lim_{x\to\infty} \frac{x+1}{xe^x} = \lim_{x\to\infty} \frac{1}{xe^x + e^x} \approx \frac{1}{\infty} = 0.$$

(b) We take logs of the limit. So

$$\begin{aligned}
\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} \\
&=1.
\end{aligned}$$

So if $\lim \ln(x+1/x)^x = 1$, then $\lim(x+1/x)^x = e^1 = e$.

(c)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right) &= \lim_{x\to 0}\left(\frac{\ln(1+x)-x}{x\ln(1+x)}\right)\\
&= \lim_{x\to 0}\left(\frac{\frac{1}{1+x}-1}{1\cdot\ln(1+x)+x\cdot\frac{1}{1+x}}\right)\\
&= \lim_{x\to 0}\left(\frac{-\frac{1}{(1+x)^2}}{\frac{1}{1+x}+1\cdot\frac{1}{1+x}+x\cdot\frac{-1}{(1+x)^2}}\right)\\
&= \lim_{x\to 0}\frac{-1}{2(1+x)-x} = \frac{-1}{2}.
\end{aligned}$$

7. (15 points) Find the following antiderivatives and definite integrals:

(a)

$$\int \cos(x)\sin(x)dx.$$

(b)

$$\int -2dx.$$

(c)

$$\int_{-1}^{1} x(2x^2+1)^5 dx.$$

Answers.

(a) We take $u = \sin(x)$, so $\frac{du}{dx} = \cos(x)$ and $du = \cos(x)dx$. Thus

$$\begin{aligned}\int \cos(x)\sin(x) &= \int \sin(x)\cos(x)dx \\ &= \int u du \\ &= u^2/2 + C \\ &= \sin^2(x)/2 + C.\end{aligned}$$

(b) $\int -2dx = -2x$.

(c) We set $u = 2x^2 + 1$, so $\frac{du}{dx} = 4x$ and $\frac{1}{4}du = xdx$. When $x = -1, u = 3$, and when $x = 1, u = 3$. Thus

$$\begin{aligned}\int_{-1}^{1} x(2x^2+1)^5 dx &= \int_3^3 \frac{u^5}{4} du \\ &= \left[\frac{u^6}{24}\right]_3^3 \\ &= 0.\end{aligned}$$

8. (5 points) Find the average of the function $f(x) = \frac{1}{\sqrt{x}}$ on the interval $[1, 4]$.

Answers. The average is given by

$$\begin{aligned} \frac{1}{4-1}\int_1^4 x^{-1/2} &= \frac{1}{3}\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_1^4 \\ &= \frac{1}{3}\left[\frac{\sqrt{x}}{\frac{1}{2}}\right]_1^4 \\ &= \frac{2}{3}. \end{aligned}$$

9. (5 points) Find the area of the planar region bounded by the curves $y = x^2$ and $y = 2x+3$.

Answers. The two curves meet at the solutions for $x^2 = 2x + 3$, which we can calculate to be equivalent to the solutions to $x^2 - 2x - 3 = 0$, or $(x-3)(x+1) = 0$ which gives us $x = -1$ and $x = 3$. Thus the area of the planar region is equal to

$$\int_{-1}^{3} 2x + 3 - x^2 dx = [x^2 + 3x - x^3/3]_{-1}^{3} = [9] - [-2 + 1/3] = 22/3.$$

10. (5 points+ 1 point Extra Credit) The odometer on Jeff's car is broken. To find the distance from Houston to his ranch out in the country, Jeff records his speedometer readings in regular intervals during a drive to the ranch. The data is presented as follows:

Time t (pm)	2:10	2:25	2:40	2:55	3:10	3:25	3:40
Speed $\sigma(t)$(mph)	55	60	65	40	40	35	50

(a) **Extra Credit**. The distance to Jeff's ranch is given by $\int_{2:10}^{3:40} \sigma(t)dt$. Explain why.

(b) Given (a), estimate the distance to Jeff's ranch using Simpson's Approximation. (Hint: You should express your time in hours, not minutes, since your speed is measured in miles per hour.)

(a) If $x(t)$ is the distance function, $\sigma(t) = x'(t)$, and so $\int \sigma(t)dt = \int x'(t)dt = x(t)$.

(b) We have Δt as 15 minutes or $\frac{1}{4}$ hours, so the Simpson's formula is

$$\frac{\frac{1}{4}}{3}\left(55 + 4(60) + 2(65) + 4(40) + 2(40) + 4(35) + 50\right).$$

This is equal to 855/12, or 71.25 miles.