Midterm Exam I
Math 101 Summer General Session 2010

Instructions: This is a **105**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Some useful identities

$\sin^2\theta + \cos^2\theta = 1$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\pi \text{ radians} = 180^\circ$$

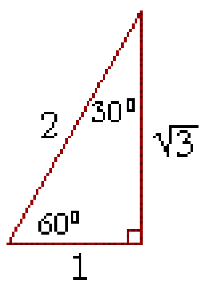
$$a^2 - b^2 = (a+b)(a-b)$$

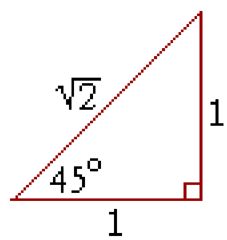
$$\log(ab) = \log a + \log b$$

$$\log(a^b) = b\log a$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\ln(x) = \log_e(x)$$





You may use the above triangles to recall certain values of sin, cos, and tan.

1. (20 points) Consider the function $f(x) = \frac{1-3x}{x}$.

   (a) Calculate $f(2)$ and $f\left(-\frac{1}{3}\right)$

   (b) What is the largest possible domain of f?

   (c) Calculate $f'(x)$ and find the slope of the tangent line at $(1, -2)$.

   (d) Write the equation of the tangent line at $(1, -2)$.

Answer.

(a) $f(2) = (1-6)/2 = -5/2$, and $f(-1/3) = (1-3(-1/3))/(-1/3) = -6$.

(b) All real numbers except zero, since x is in the denominator.

(c) $f(x) = \frac{1}{x} - 3 = x^{-1} + 3$, so by the power rule $f'(x) = -x^{-2}$. The slope of the tangent line is $f'(1) = -1$.

(d) We have a point $(1, -2)$ and a slope -1. The line has the form $y = -x + c$, and substituting the point $(1, -2)$ in we have $c = -1$. Thus the equation of the tangent line is $y = -x - 1$.

2. (15 points) Calculate the following limits:

(a)

$$\lim_{x\to 0} \frac{\tan x}{x}.$$

(b)

$$\lim_{x\to 0} \frac{\sin^2 x}{2x}.$$

(c)

$$\lim_{x\to 3} (2x^2 - 3).$$

Answer.

(a)

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{x} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \\ &= 1 \cdot \frac{1}{\cos 0} \\ &= 1 \cdot 1 = 1.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2 x}{2x} &= \lim_{x\to 0} \frac{\sin x}{x} \frac{\sin x}{2} \\ &= 1 \cdot \frac{\sin 0}{2} \\ &= 1 \cdot 0 = 0\end{aligned}$$

(c) $\lim_{x\to 3}(2x^2 - 3) = 2(3)^2 - 3 = 15.$

3. (10 points) Consider the graph $y = \cos 2x$

   (a) What is the derivative $\dfrac{dy}{dx}$?

   (b) Find one point of the graph where the tangent line is horizontal.

Answers.

(a) $\dfrac{dy}{dx} = -\sin(2x) \cdot 2 = -2\sin(2x)$, by the chain rule.

(b) $-2\sin(2x) = 0$ when $x = 0$, and when $x = 0$, $y = \cos(0) = 1$. Thus the point we want is $(0, 1)$.

4. (20 points) Find the derivative $\frac{dy}{dx}$ when

(a) $y = 2$

(b) $y = \frac{1}{\sqrt{x}}$.

(c) $y = 4x^3 + 2x^2 - 3x + 1$.

(d) $y = \sin x \cos x$.

Answers.

(a) $\frac{dy}{dx} = 0$.

(b) $y = x^{-1/2}$, so $\frac{dy}{dx} = (-1/2)x^{-3/2}$ by the power rule.

(c) $\frac{dy}{dx} = 12x^2 + 4x - 3$ by repeated application of the power rule.

(d) $\frac{dy}{dx} = \cos x \cos x + \sin x(-\sin x)$ by the product rule.

5. (15 points) Find the derivative $f'(x)$ when

(a) $f(x) = \ln(\ln x)$.

(b) $f(x) = e^{2x}$.

(c) $f(x) = \frac{e^x}{\cos x}$.

Answers.

(a) $f'(x) = \frac{\ln'(x)}{\ln x} = \frac{1}{x} \cdot \ln x$.

(b) $f'(x) = e^{2x} \cdot 2$, since $\dfrac{d2x}{dx} = 2$.

(c) $f'(x) = \frac{e^x \cos x - e^x(-\sin(x))}{\cos(x)^2} - \frac{e^x(\cos x + \sin x)}{\cos(x)^2}$

6. (5 points) A bug is traveling on the x-axis. Its position $x(t)$ at time t is given by the function $x(t) = -16t^2 + 32t + 25$. Find the bug's position x when its velocity is zero. (Hint: velocity is found by differentiating position with respect to time).

   Answers The velocity function $x'(t)$ is $-32t + 32$. So when $x'(t) = 0$, $-32t + 32 = 0$ and so $t = 1$. Thus the time when the bug's velocity is zero is at $t = 1$. So the bug's position at that time will be $x(1) = -14 + 32 + 25 = 41$

7. (10 points) Let x, y be two non-negative numbers such that $x + 2y = 12$. What is the largest possible product of x and y?

Answers. Let us express the product as $f(x) = xy$. We know that $2y = 12 - x$, so $y = \frac{12-x}{2}$. Thus $f(x) = x\frac{12-x}{2} = \frac{12x-x^2}{2}$. But then $f'(x) = 6 - x$, and so we have a critical point when $x = 6$. But clearly x lies in between $x = 0$ and $x = 12$. We have $f(0) = 0$ and $f(12) = 0$, and so $f(6) = 18$ must be the maximal value.

8. (5 points) Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = x^x$.

Answers.

$$
\begin{aligned}
\ln y =& \ln(x^x) \\
=& x \ln x \\
\frac{d \ln y}{dx} =& \frac{dx \ln x}{dx} \\
\frac{\frac{dy}{dx}}{y} =& \ln x + x(\frac{1}{x}) \\
\frac{dy}{dx} =& (\ln(x) + 1)y \\
=& (\ln(x) + 1)x^x
\end{aligned}
$$