Practice Final Exam
Math 101 Summer General Session 2010

Instructions: This is a **105**-minute exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem. Attach extra paper if you need more space.

Write your name:

Write out the Honor Pledge: "On my honor, I have neither given nor received any unauthorized aid on this exam."

Signature:

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

Some useful identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\pi \text{ radians} = 180°$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\log(ab) = \log a + \log b$$

$$\log(a^b) = b\log a$$

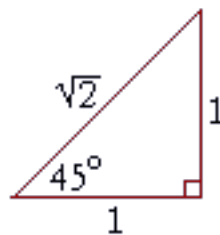$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\ln(x) = \log_e(x)$$

$$\log_a(b) = \frac{\ln b}{\ln a}$$

$$a^b = e^{b\ln a}.$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Useful Formulas

The volume of a planar region under the curve $y = f(x)$ rotated around the x-axis is given by

$$\int_a^b \pi f(x)^2 dx.$$

The volume of a planar region under the curve $y = f(x)$ rotated around the y-axis is given by

$$\int_a^b 2\pi x f(x) dx.$$

The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

The surface area of a planar region under the curve $y = f(x)$ rotated around the x-axis is given by

$$\int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx.$$

The surface area of a planar region under the curve $y = f(x)$ rotated around the y-axis is given by

$$\int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} dx.$$

The coordinates of the centroid $(\bar{x}, \bar{y})$ of a planar region under the curve $y = f(x)$ with area A is given by

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx,$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 dx.$$

1. (20 points) Find $\dfrac{dy}{dx}$ when

    (a) $y = \pi$.

    (b) $y = e^{-x} \sin x$.

    (c) $y = \dfrac{x}{\cos x}$

    (d) $y = 10^{1/x}$.

    Answers.

    (a) π is a constant, so $\dfrac{dy}{dx} = 0$.

    (b) $\dfrac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$.

    (c) $\dfrac{dy}{dx} = \dfrac{1 \cdot \cos x - x(-\sin x)}{(\cos x)^2} = \dfrac{\cos x + x \sin x}{(\cos x)^2}$

    (d) $y = 10^{1/x} = e^{(\ln 10)(1/x)}$. Thus $\dfrac{dy}{dx} = (-\ln(10)/x^2)e^{\ln(10)/x}$, by the chain rule.

2. (20 points) Evaluate the following integrals:

(a) $\int_0^1 x\,dx$.

(b) $\int \frac{\sin x}{\cos x}\,dx$.

(c) $\int 3^{2x}\,dx$.

(d) $\int \frac{1}{\sqrt{4+x^2}}\,dx$.

Answers.

(a) $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.

(b) Make the u-substitution $u = \cos x$. We then have $\dfrac{du}{dx} = -\sin x$, and $\sin x\,dx = -du$.

   Thus $\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}du = \frac{1}{u^2} + C = \frac{1}{\cos^2 x} + C$.

(c) $\int 3^{2x}\,dx = \int 9^x\,dx = \int e^{x\ln 9}\,dx = \frac{e^{x\ln 9}}{\ln 9} + C = \frac{9^x}{\ln 9} + C$.

(d) $\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{1/\sqrt{4}}{\sqrt{1+\frac{x^2}{4}}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}}\,dx$. Taking $u = x/2$, we have $dx = 2du$ and so

   the integral is equal to $\int \frac{1}{\sqrt{1+u^2}}\,du$ or $\operatorname{arcsinh}(u)+C$, which is equal to $\operatorname{arcsinh}(x/2)+C$.

3. (10 points)Use l'Hôpital's rule to find the following limits:

(a)
$$\lim_{x\to\infty} \frac{\ln x}{e^x}.$$

(b)
$$\lim_{x\to 0} x^{\sin x}.$$

Answers.

(a)
$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} \approx \frac{0}{\infty} = 0.$$

(b) We set $y = x^{\sin x}$. Then

$$
\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \sin x \ln x \\
&= \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{-x\cos x} \\
&= \lim_{x\to 0} \frac{2\sin x \cos x}{-\cos x + x\sin x} \\
&\approx \frac{0}{-1} = 0.
\end{aligned}
$$

Since $\lim \ln y = 0$, $\lim y = e^0 = 1$.

4. (5 points) The width of a rectangle is half its length. At what rate is its area increasing if its width is 10cm and is increasing at a rate of 0.5 cm per second?

Answers. Our rectangle has width x and length $2x$. Thus we may express the area as $y = x(2x) = 2x^2$. The width is increasing at a rate of 0.5 cm per second, this is equivalent to saying $\dfrac{dx}{dt} = 0.5$. The rate that the area increases can be expressed as $\dfrac{dy}{dt}$. By the chain rule we have

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\
&= (4x)(0.5) \\
&= 2x
\end{aligned}
$$

But $x = 10$, so the rate the area is increasing must be 20.

5. (5 points) Prove that the function $x^7 + x^5 + x^3 + 1 = 0$ has exactly one real solution.

Answers. Let $f(x) = x^7 + x^5 + x^3 + 1$. We can check that $f(-1)$ is negative and $f(1)$ is positive, thus there must be at least one solution between $x = -1$ and $x = 1$. We can also determine that $f'(x) = 7x^6 + 5x^4 + 3x^2$. This is a sum of squares, and so it is always non-negative. But this implies that $f(x)$ never decreases, and so it can only have one solution.

6. (10 points) A particle is moving along a line with velocity function $v(t) = 2t + 10$. From the time $t = 1$ to the time $t = 5$, calculate the particle's

(a) net distance traveled

(b) total distance traveled.

Answers. The function $v(t) = 2t + 10$ is always positive in the interval $[1, 5]$, and so the net distance is equal to the total distance. This value is calculated as

$$\int_1^5 v(t)dt = \int_1^5 2t + 10 dt$$
$$= \left[t^2 + 10t \right]_1^5$$
$$= 75 - 11$$
$$= 64.$$

So both the net and total distances are 64.

7. (10 points) Calculate the following volumes.

    (a) The region bounded by $y = x^2$ and $x = y^2$, rotated around the x-axis.

    (b) The region bounded by $y = x^2, y = 0, x = 1, x = -1$, rotated around the line $x = 2$.

Answers

    (a) We may write the two graphs as $y = x^2$ and $y = \sqrt{x}$. To determine the bounds of our integral, we solve for $x^2 = \sqrt{x}$, which is true only when $x = 0$ or $x = 1$. Thus the volume is given by

$$\int_0^1 \pi((\sqrt{x})^2 - (x^2)^2)dx = \int_0^1 \pi(x - x^4)dx$$
$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$
$$= \frac{3\pi}{10}.$$

    (b) The line $x = 2$ is an offset y-axis. So the volume is determined as

$$\int_{-1}^1 2\pi(2 - x)(x^2)dx = 2\pi \int_{-1}^1 2x^2 - x^3 dx$$
$$= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_{-1}^1$$
$$= 2\pi\left(\left[\frac{5}{12}\right] - \left[\frac{-11}{12}\right]\right)$$
$$= \frac{8}{3}\pi.$$

    Note. You may want to write down the integral as $\int_{-1}^1 2\pi(x - 2)(x^2)dx$ instead: this will give you the answer, but with wrong sign (your answer will be $-8\pi/3$). This is alright, because the sign doesn't really matter in these volume problems. I feel that the advantage of presenting the integral as $\int_{-1}^1 2\pi(x - 2)(x^2)dx$ is that you can remember the simple rule that a rotation about the line $x = c$ has volume $\int 2\pi(x - c)f(x)dx$: this is true regardless of whether c is positive or negative.

10

8. (10 points)

(a) Set up and simplify the integral that gives the surface area of revolution generated by rotation of the arc $y = x^2$, for $0 \leq x \leq 4$ around the y-axis. **Do not evaluate the integral!**

(b) Find the length of the smooth arc $y = \frac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = 1$.

Answers.

(a) We have

$$\frac{dy}{dx} = 2x, \quad \left(\frac{dy}{dx}\right)^2 = 4x^2$$

Thus the surface area is equal to

$$\int_0^4 2\pi x \sqrt{1 + 4x^2} \, dx.$$

(b) We have

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(e^x - e^{-x})^2}{4}$$

$$\frac{e^{2x} + e^{-2x} - 2}{4}$$

$$\text{Arclength} = \int_0^1 \sqrt{1 + \frac{e^{2x} + e^{-2x} - 2}{4}} \, dx$$

$$= \int_0^1 \sqrt{\frac{2 + e^{2x} + e^{-2x}}{4}} \, dx$$

$$= \frac{1}{2} \int_0^1 \sqrt{(e^x + e^{-x})^2} \, dx$$

$$= \frac{1}{2} \int_0^1 (e^x + e^{-x}) \, dx$$

$$= \frac{1}{2} \left[e^x - e^{-x} \right]_0^1 = \frac{e - e^{-1}}{2}.$$

9. (10 points) Find the centroid of the planar region bounded by $y = x^2, y = 18 - x^2$.

Answers. We can immediately see that $\bar{x} = 0$ by symmetry. Let us determine the area of the planar region. The two graphs intersect at the x-values where $x^2 = 18 - x^2$, or $x^2 = 9$. Thus the points of intersection of these two graphs are at $x = -3, x = 3$. Thus we have the area

$$A = \int_{-3}^{3} (18 - x^2) - x^2 dx = \left[18x - \frac{2x^3}{3} \right]_{-3}^{3} = 72.$$

We then know that

$$\begin{aligned}
\bar{y} &= \frac{1}{72} \int_{-3}^{3} \frac{1}{2} ((18 - x^2)^2 - (x^2)^2) dx \\
&= \frac{1}{144} \int_{-3}^{3} (324 - 36x^2 + x^4 - x^4) dx \\
&= \frac{1}{144} \int_{-3}^{3} 324 - 36x^2 dx \\
&= \frac{1}{144} \left[324x - 12x^3 \right]_{-3}^{3} \\
&= 9.
\end{aligned}$$