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HW #9

(1.) F: not a distribution: if $\varphi(x) = \sum_{n=0}^{\infty} 2^n x^n$ near $x=0$ (this series converges for $|x| < \frac{1}{2}$), then $\varphi^{(n)}(0) = n!\,2^n$, so the series "defining" F diverges.
Such a φ can be extended to a test function $\varphi_0 \in C_c^\infty$, $\varphi_0 = \varphi$ near $x=0$.

G: $G \in \mathcal{D}'$: If $\operatorname{supp}\varphi \subset K \subset [-N,N]$, then $|\langle G,\varphi\rangle| \le 2^N \sum_{n=0}^{N} \|\varphi^{(n)}\|_\infty$.

H: not a distribution: this is not linear in φ.

(2.) Suppose that $F_n \to F$ in $\mathcal{D}'$. Then $\langle F_n', \varphi\rangle = -\langle F_n, \varphi'\rangle \to -\langle F, \varphi'\rangle = \langle F', \varphi\rangle$, so $F_n' \to F'$ in $\mathcal{D}'$ also.

Clearly $\frac{1}{n} e^{inx} \to 0$ in $\mathcal{D}'$. By taking $k+1$ derivatives, this implies that $n^k e^{inx} \to 0$ in $\mathcal{D}'$.

[This can be shown directly, using the fact that $\hat{\varphi} \in \mathcal{S}$ if $\varphi \in C_c^\infty$.]

(3.) (a) $\left|\frac{1}{h}(\varphi(x+h) - \varphi(x)) - \varphi'(x)\right| = \left|\frac{1}{h}\int_x^{x+h} \underbrace{(\varphi'(t) - \varphi'(x))}_{|\cdots| \le \|\varphi''\|_\infty h} dt\right| \le \|\varphi''\|_\infty h$

(say $h > 0$)

(b) By the definition of $\tau_h F$, F', we have that
$$\left\langle \frac{1}{h}(\tau_h F - F) - F', \varphi\right\rangle = \left\langle F, \frac{\tau_{-h}\varphi - \varphi}{h} + \varphi'\right\rangle$$

By part (a), applied to $\varphi, \varphi', \varphi'', \dots$, we obtain that for any $k \geq 0$, $\left\|\frac{d^k}{dx^k}\left(\frac{\tau_{-h}\varphi - \varphi}{h} + \varphi'\right)\right\|_\infty \to 0$ as $h \to 0$, and since $|\langle F, \psi\rangle| \le C \sum_{n=0}^{N} \|\psi^{(n)}\|_\infty$, it follows that $\left\langle F, \frac{\tau_{-h}\varphi - \varphi}{h} + \varphi'\right\rangle \to 0$, as desired.

(4.) Recall that $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$. Also, observe that $\{Y \cap B : B \in \mathcal{B}_X\}$ is a σ-algebra on Y, and $\mathcal{T}_Y \subset \{\cdots\}$. Thus $\mathcal{B}_Y \subset \{\cdots\}$.
Conversely, let $\mathcal{E} = \{E \in \mathcal{B}_X : Y \cap E \in \mathcal{B}_Y\}$. This is a σ-algebra on X.
Indeed, if $E \in \mathcal{E}$, then $Y \cap E^c = Y \setminus (Y \cap E) \in \mathcal{B}_Y$, so $E^c \in \mathcal{E}$. Similarly, if $E_j \in \mathcal{E}$, then $Y \cap \bigcup E_j = \bigcup (Y \cap E_j) \in \mathcal{B}_Y$, so $\bigcup E_j \in \mathcal{E}$.
Since $\mathcal{T}_X \subset \mathcal{E}$, it follows that $\mathcal{E} = \mathcal{B}_X$, and this establishes $\{\cdots\} \subset \mathcal{B}_Y$.