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HW #8

(1.) (a) $|\langle f_n,\varphi\rangle - \langle f,\varphi\rangle| = |\int (f_n - f)\varphi\,dx| \underset{\text{Hölder}}{\leq} \|f_n - f\|_p \|\varphi\|_q \to 0$ $\}$ if $\|f_n-f\|_p \to 0$

(b) $\int (f_n - f)\varphi\,dx \to 0$ by DC: $|(f_n-f)\varphi| \leq \chi_K(|g|+|f|)\|\varphi\|_\infty \in L^1$

$\}K = \operatorname{supp}\varphi$

(c) $f_n(x) = \begin{cases} n/2 & |x| \leq \frac{1}{n},\ x \neq 0 \\ 0 & |x| > 1/n \text{ or } x = 0 \end{cases}$

Then $f_n(x) \to 0$ for all $x \in \mathbb{R}$, but $f_n \to \delta$ in $\mathcal{D}'$ (why?).

(2.) $\langle \partial_j(\underset{C^\infty}{\psi} F), \varphi\rangle = -\langle \psi F, \partial_j\varphi\rangle = -\langle F, \psi\partial_j\varphi\rangle = -\langle F, \partial_j(\psi\varphi)\rangle + \langle F, \varphi\partial_j\psi\rangle$

$= \langle \psi\partial_j F, \varphi\rangle + \langle (\partial_j\psi)F, \varphi\rangle$, so $\partial_j(\psi F) = \psi\partial_j F + (\partial_j\psi)F$

(3.) $\langle \delta^{(j)}, \varphi\rangle = (-1)^j \varphi^{(j)}(0)$, so $\langle \psi\delta^{(k)}, \varphi\rangle = (-1)^k (\psi\varphi)^{(k)}(0)$

$= (-1)^k \sum_{j=0}^{k} \binom{k}{j} \psi^{(j)}(0) \underbrace{\varphi^{(k-j)}(0)}_{=(-1)^{k-j}\langle \delta^{(k-j)}, \varphi\rangle} = \langle \sum_{j=0}^{k} \binom{k}{j} \psi^{(j)}(0)\,\delta^{(k-j)}, \varphi\rangle$, as claimed

$\}$ prove this version of the product rule if you're not familiar with it!

(5.) $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\sum_{j=0}^{m} \int_{x_j}^{x_{j+1}} f(x)\varphi'(x)\,dx$ and

$\}$ $x_0 := -\infty$, $x_{m+1} := \infty$

$$-\int_{x_j}^{x_{j+1}} f\varphi'\,dx \underset{\text{int. by parts}}{=} -f\varphi\Big|_{x_j}^{x_{j+1}} + \int_{x_j}^{x_{j+1}} f'\varphi\,dx = -f(x_{j+1}-)\,\varphi(x_{j+1}) + f(x_j+)\,\varphi(x_j) + \int_{x_j}^{x_{j+1}} f'\varphi\,dx;$$

also $-\int_{-\infty}^{x_1} f\varphi'\,dx = -f(x_1-)\varphi(x_1) + \int_{-\infty}^{x_1} f'\varphi\,dx$ since φ has compact support,

and similarly $-\int_{x_m}^{\infty} f\varphi'\,dx = f(x_m+)\varphi(x_m) + \int_{x_m}^{\infty} f'\varphi\,dx$.

This gives the claim after rearranging.