5463 HW#7

(1.) $\hat{f}(\xi) = \int_{-1}^{1} e^{-2\pi i \xi x} dx = \frac{1}{-2\pi i \xi}(e^{-2\pi i \xi} - e^{2\pi i \xi}) = \frac{\sin 2\pi\xi}{\pi \xi}$ { this final formula also works for $\xi = 0$, if we interpret $\left.\frac{\sin 2\pi\xi}{\pi\xi}\right|_{\xi=0} = \lim_{\xi\to 0} \dots = 2$

$$\hat{g}(\xi) = \underbrace{\int_{-1}^{0} (1+x) e^{-2\pi i \xi x} dx}_{= \int_0^1 (1-x) e^{2\pi i \xi x} dx \text{ int. by parts } (x \mapsto -x)} + \int_{0}^{1} (1-x) e^{-2\pi i \xi x} dx \rightsquigarrow \hat{g}(\xi) = 2\int_0^1 (1-x)\cos 2\pi\xi x \, dx$$

$$= \frac{1}{\pi\xi}(1-x)\sin 2\pi\xi x \Big|_0^1 + \frac{1}{\pi\xi}\int_0^1 \sin 2\pi\xi x \, dx = -\frac{1}{2\pi^2\xi^2}\cos 2\pi\xi x \Big|_0^1 = \frac{1-\cos 2\pi\xi}{2\pi^2\xi^2} = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2$$

(2.) By Fourier inversion and the assumptions on $f, \hat{f}$,

$$f(x+h) - f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} (e^{2\pi i h \xi} - 1) d\xi \quad (\text{for } \underline{\text{all}}\ x, h)$$

$$\underset{\text{hint}}{=} \int_{-1/|h|}^{1/|h|} \dots + \int_{|\xi| > 1/|h|} \equiv I_1 + I_2$$

Now $|e^{2\pi i h\xi} - 1| = \left| i \int_0^{2\pi \xi h} e^{it} dt \right| \lesssim |\xi h|$, so $|I_1| \lesssim \int_{-1/|h|}^{1/|h|} \frac{|\xi h|}{(1+|\xi|)^{1+\alpha}} d\xi$

$\le |h| \int_{-1/|h|}^{1/|h|} \frac{d\xi}{(1+|\xi|)^{\alpha}} \lesssim |h| \left(\frac{1}{|h|}\right)^{1-\alpha} = |h|^{\alpha}$, as desired. { why is it OK to write this instead of $1+|\xi|^{1+\alpha}$

Similarly, $|I_2| \lesssim \int_{|\xi| > 1/|h|} |\xi|^{-1-\alpha} d\xi \lesssim |h|^{\alpha}$.

8.2.8. First of all, $(f*g)(x)$ is defined pointwise by Prop. 8.8, and so is $h*g$ ($h \equiv \partial_j f$). Assume that $j=1$, for convenience.

We want to show that $\frac{1}{h}((f*g)(x+he_1) - (f*g)(x)) - (h*g)(x) \xrightarrow[h\to 0]{} 0$.

$$|LHS| = \left| \int \Big(\underbrace{\frac{f(x+he_1-y) - f(x-y)}{h} - h(x-y)}_{= \frac{1}{h}(\tau_{-he_1} f - f) - h \,|_{x-y}} \Big) g(y) dy \right| \underset{\text{Hölder}}{\le} \| \tfrac{1}{h}(\tau_{-he_1} f - f) - h \|_p \|g\|_q$$

Hölder, $\|F(x-\cdot)\|_p = \|F(\cdot)\|_p$

This does go to zero, by the definition of h.

8.2.9: Claim: $f \in AC_{loc}(\mathbb{R})$, $f' \in L^p \iff f$ has an L^p derivative

"$\Rightarrow$": By Thm 3.35, $\frac{f(x+y) - f(x)}{y} - f'(x) \left(= \frac{1}{y}(\tau_{-y} f - f) - f' \Big|_x \right) = \frac{1}{y}\int_0^y (f'(x+t) - f'(x)) dt$,

so $\| \dots \|_p = \frac{1}{y} \| \int_0^y (f'(\cdot - t) - f'(\cdot)) dt \|_p \underset{\text{Minkowski}}{\le} \frac{1}{y} \int_0^y \| \tau_{-t} f' - f' \|_p dt$ { I'm assuming $y \ge 0$ here

{ the variable is x here!

Now $t \mapsto \| \tau_{-t} f' - f' \|_p$ is continuous (P8.5), so $\frac{1}{y}\int_0^y \| \dots \|_p dt \xrightarrow[y\to 0+]{} \| \tau_0 f' - f' \|_p = 0$, as desired.

"$\Leftarrow$": As instructed, pick $g \in C_c^\infty(\mathbb{R})$, $\int g = 1$. Then $f * g_t \to f$ in L^p and thus also pointwise a.e. on a subsequence $t_n \to 0+$. Fix $a \in \mathbb{R}$ s.t. $(f*g_{t_n})(a) \to f(a)$. Denote the L^p derivative of f by h. Recall that $f * g_t \in C^\infty$. Thus $(f*g_t)(x) = (f*g_t)(a) + \int_a^x (f*g_t)'(s) ds$ $\rightarrow$

Exercise 8 says that $(f*g_t)' = h*g_t$. Now send $t \to 0+$, along the sequence t_n. Since $h*g_t \to h$ in L^p, we obtain that for a.e. x (for those for which $(f*g_{t_n})(x) \to f(x)$)

$$f(x) = f(a) + \int_0^a h(s)\,ds. \quad (*)$$

This says that $f \in AC$ (after modifying on a null set, so that $(*)$ holds for all x) and $f' = h$ (so in particular, $f' \in L^p$).