5463 HW#6 (1)

(1.) (a) $g \geq 0$, so MC shows that $\int_0^1 f(x)dx = \sum_{n=1}^{\infty} 2^{-n} \underbrace{\int_0^1 g(x-r_n)dx}_{\leq \int_0^1 x^{-1/2}dx = 2} \leq 2$. (*)

In particular, the series converges a.e.; f is measurable by Prop. 2.11(b), and $f \in L^1$ by (*) ($f \geq 0$!).

(b) Let $E^c = \left(\bigcup_{n\geq 1} [r_n - \varepsilon \cdot 2^{-n-1}, r_n + \varepsilon \cdot 2^{-n-1}]\right) \cap (0,1)$. Then $m(E^c) \leq \sum_{n=1}^{\infty} \varepsilon \cdot 2^{-n} = \varepsilon$

and if $x \in E$, then $2^{-n} g(x-r_n) \leq 2^{-n}(\varepsilon 2^{-n-1})^{-1/2} = \varepsilon^{-1/2} 2^{1/2} \cdot 2^{-n/2}$.

So if $\delta > 0$ is given, we can find $N \in \mathbb{N}$ s.t. $\sum_{n>N} 2^{-n} g(x-r_n) < \delta$ for all $x \in E$ and then $\sigma > 0$ s.t. also $\left|\sum_{n=1}^{N} 2^{-n} g(x-r_n) - \sum_{n=1}^{N} 2^{-n} g(x'-r_n)\right| < \delta$ if $x' \in E$ also and $|x-x'| < \sigma$. } the function $\sum_{n=1}^{N} 2^{-n} g(x-r_n)$ is clearly continuous on E!

Thus $|f(x) - f(x')| < 3\delta$ if $\substack{|x-x'| < \sigma \\ x' \in E}$ (note that we fix $x \in E$ here).

8.1.1: First of all, we need to interpret this correctly:

$\partial^\alpha (fg) = \sum_{\substack{0 \leq \beta \leq \alpha \\ \gamma := \alpha - \beta}} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g)$; I write $\beta \leq \alpha$ for $\beta_1 \leq \alpha_1, \ldots, \beta_n \leq \alpha_n$.

By induction on $|\alpha|$: $|\alpha| = 1$, say $\alpha = (1,0,\ldots,0)$: $\partial_1(fg) = \partial_1 f g + f \partial_1 g$ by the traditional product rule, which is also what the formula gives ($\beta = (0,0,\ldots,0)$ or $\beta = (1,0,\ldots,0)$ in the sum).

$|\alpha| - 1 \to |\alpha|$: Say $\alpha_1 \geq 1$. By the IH, $\partial^\alpha(fg) = \partial_1 \partial^{\alpha - e_1}(fg)$

$$= \sum_{\substack{0 \leq \beta \leq \alpha - e_1 \\ \beta + \gamma = \alpha - e_1}} \frac{(\alpha - e_1)!}{\beta!\gamma!} \left((\partial^{\beta + e_1} f)(\partial^\gamma g) + (\partial^\beta f)(\partial^{\gamma + e_1} g)\right)$$

$$= \sum_{\substack{e_1 \leq \beta \leq \alpha \\ \beta + \gamma = \alpha}} \frac{(\alpha - e_1)!}{(\beta - e_1)!\gamma!} (\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{0 \leq \beta \leq \alpha - e_1 \\ \beta + \gamma = \alpha}} \frac{(\alpha - e_1)!}{\beta!(\gamma - e_1)!} (\partial^\beta f)(\partial^\gamma g)$$

} $\tilde{\beta} = \beta + e_1$; $\tilde{\beta} \rightsquigarrow \beta$ } $\tilde{\gamma} = \gamma + e_1$; $\tilde{\gamma} \rightsquigarrow \gamma$

Let's now look at these terms for a fixed β, $0 \leq \beta \leq \alpha$:

(1) $\beta_1 = 0$: Only the second sum contributes, and $\gamma_1 = \alpha_1 \rightsquigarrow \frac{\alpha_1 - 1}{\gamma_1 - 1} = 1 = \frac{\alpha_1}{\gamma_1} \rightsquigarrow$ $\frac{(\alpha - e_1)!}{\beta!(\gamma - e_1)!} = \frac{\alpha!}{\beta!\gamma!}$, as desired.

(2) $\beta_1 = \alpha_1$: Similar; (3) $1 \leq \beta_1 \leq \alpha_1 - 1$: $\frac{(\alpha - e_1)!}{(\beta - e_1)!\gamma!} + \frac{(\alpha - e_1)!}{\beta!(\gamma - e_1)!}$

$= (\alpha - e_1)! \frac{\beta_1 + \gamma_1}{\beta!\gamma!} = \frac{(\alpha - e_1)!\alpha_1}{\beta!\gamma!} = \frac{\alpha!}{\beta!\gamma!}$

You can also try to give a (less formal) combinatorial argument.

The first identity is now immediate (from the product rule). ②

To establish the second, we can again use induction on $|\alpha|$:

$|\alpha|=1$, say $\alpha=e_1$: $x^\beta \partial_1 f = \partial_1(x^\beta f) - \beta_1 x^{\beta-e_1} f$

$|\alpha|-1 \rightsquigarrow |\alpha|$; say $\alpha_1 \geq 1$: $x^\beta \partial^\alpha f = x^\beta \partial_1 \partial^{\alpha-e_1} f = \partial_1 x^\beta \partial^{\alpha-e_1} f - \beta_1 x^{\beta-e_1} \partial^{\alpha-e_1} f$

$$\underset{IH}{=} \underbrace{\partial_1 \partial^{\alpha-e_1}}_{\partial^\alpha} x^\beta f + \partial_1 \sum_{\substack{|\gamma|<|\alpha|-1 \\ |\delta|<|\beta|}} c'_{\gamma\delta} \partial^\gamma (x^\delta f) - \beta_1 \partial^{\alpha-e_1}(x^{\beta-e_1} f) - \sum_{\substack{|\gamma|<|\alpha|-1 \\ |\delta|<|\beta|-1}} c''_{\gamma\delta} \partial^\gamma (x^\delta f)$$

This is of the desired form, after rearranging (slightly).

8.1.2: (a) I'll do this with a combinatorial argument: How many ways are there to produce $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, for a given α: Need to pick x_1 from α_1 out of k factors ($\rightsquigarrow \binom{k}{\alpha_1}$ possibilities), then x_2 from α_2 out of the remaining $k-\alpha_1$ factors ($\rightsquigarrow \binom{k-\alpha_1}{\alpha_2}$) etc. So the coefficient of x^α is

$$\binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2}\cdots\binom{k-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha_1!\cancel{(k-\alpha_1)!}} \frac{\cancel{(k-\alpha_1)!}}{\alpha_2!\cancel{(k-\alpha_1-\alpha_2)!}} \cdots \frac{\cancel{(k-\alpha_1-\cdots-\alpha_{n-1})!}}{\alpha_n!\underbrace{(k-\alpha_1-\cdots-\alpha_n)!}_{=0!=1}}$$

$$= \frac{k!}{\alpha_1!\cdots\alpha_n!} = \frac{k!}{\alpha!}$$

(b) This just follows by multiplying the formulae $(x_j+y_j)^{\alpha_j} = \sum_{\substack{0\leq\beta_j\leq\alpha_j \\ \beta_j+\gamma_j=\alpha_j}} \frac{\alpha_j!}{\beta_j!\gamma_j!} x_j^{\beta_j} y_j^{\gamma_j}$.

8.1.3: (a) By induction on $k\geq 0$: $k=0$: ✓

$k\to k+1$: $\eta^{(k+1)} = \frac{d}{dt}\eta^{(k)} \underset{IH}{=} \frac{d}{dt}\left(P_k(1/t)e^{-1/t}\right) = \underbrace{-\tfrac{1}{t^2}P_k'(1/t)}_{\deg(\ldots)=2k+1} e^{-1/t} \underbrace{-\tfrac{1}{t^2}P_k(1/t)}_{\deg(\ldots)=2k+2} e^{-1/t}$

(b) Again by induction on $k\geq 0$: $k=0$:

$$\frac{\eta(h)-\eta(0)}{h} \underset{\text{for } h>0}{=} \tfrac{1}{h}e^{-1/h} \to 0 \quad (h\to 0+)$$

$h<0 \rightsquigarrow \eta(h)=0$ is trivial

$k\to k+1$: $\frac{1}{h}(\eta^{(k)}(h)-\eta^{(k)}(0)) \underset{IH,(a)}{=} \frac{1}{h}P_k(h)e^{-1/h} \to 0 \quad (h\to 0+)$