(1) (a) ✓

(b) It's clear that f is continuous at all $x \in X_0$ (why?).

Since $X \setminus \operatorname{supp} f_0$ is a neighborhood of ∞ and $f=0$ here, we also have continuity at ∞.

(c) Claim: $\mu_0(E) = \mu(E)$ $(E \in \mathcal{B}_{X_0} \overset{\text{(why?)}}{\rightsquigarrow} E \in \mathcal{B}_X)$

$I_0(f_0) = I(f) = \int_X f\,d\mu \underset{\{f(\infty)=0}{=} \int_{X_0} f\,d\mu$, so the claim will follow if we can show that $\mu_0 \equiv \mu|_{X_0}$ is a Radon measure. Outer regularity: let $E \in \mathcal{B}_{X_0}$, $\varepsilon > 0$. Then (μ = R measure on X!) $\mu(\tilde{\mathcal{U}}) \le \mu(E) + \varepsilon$ for some open (in X) $\tilde{\mathcal{U}} \supseteq E$. Let $\mathcal{U} = \tilde{\mathcal{U}} \setminus \{\infty\}$ = open in X_0. Then still $\mathcal{U} \supset E$, $\underset{\mu_0(\mathcal{U})}{\mu(\mathcal{U})} \le \underset{\mu_0(E)}{\mu(E)} + \varepsilon$

Inner regularity on open sets is similar (slightly easier).

(d) By (c), if we could extend, then the representing measure μ_0 of $I_0 = I|_{C_c(X_0)}$ is finite (why?). In general, this need not hold; for example, $X_0 = \mathbb{R}$, $I_0(f_0) = \int_{-\infty}^{\infty} f_0(x)\,dx$.

(e) For $f_0 \in C_0(X_0)$, we define $f \in C(X)$ as in part (b). Conversely, if $f \in C(X)$, then there is a unique $c \in \mathbb{C}$ s.t. $(f-c)|_{X_0} \in C_0(X_0)$; we have to take $c = f(\infty)$. In particular, if $f \in C(X)$ came from $f_0 \in C_0(X_0)$ by extending, then $c = 0$.

Now if I extends I_0, then $I(f) = I(f-c) + I(c) = I_0((f-c)_0) + c\,I(1)$.

$\equiv I_0((f-c)_0) + \alpha c \quad \{\alpha = I(1) \ge 0$

Conversely, this formula defines an extension I of I_0 for any $\alpha \ge 0$

Then $I_0(f_0) = I(f) = \int_X f\,d\mu = \int_{X_0} f\,d\mu$, so again $\mu_0 = \mu|_{X_0}$.

$\{C_0(X_0) \rightsquigarrow c = 0$ for f

As before, this also shows that $\mu_0(X_0) < \infty$ for any μ_0 representing an I_0.

So we can obtain μ from μ_0 as follows: $I(f) = I_0((f - f(\infty))_0) + \alpha f(\infty)$

$= \int_{X_0} (f - f(\infty))\,d\mu_0 + \alpha f(\infty) = \int_{X_0} f\,d\mu_0 + f(\infty)(\alpha - \mu_0(X_0))$

This equals $\int_X f\,d\mu$ for $\mu(E) = \mu_0(E \cap X_0) + (\alpha - \mu_0(X_0))\delta_\infty(E)$

$\equiv \mu_0(E \cap X_0) + \beta\delta_\infty(E)$, and here $\beta = \alpha - \mu_0(X_0) \ge 0$ because $\alpha = I(1) \ge \mu(X_0) = \mu_0(X_0)$

7.1.3: X is compact, so $\mu(X) < \infty$. If $\mu(\{x\}) = 0$ for ~~some~~ $x \ne \infty$, then $x \notin \operatorname{supp}\mu$, since $\{x\}$ is open. There are at most countably many $x \ne \infty$ with $\mu(\{x\}) > 0$ (otherwise $\mu(X) = \infty$ ↯).

7.1.4: (a) f is bounded, so $f^{-1}([a,\infty)) = f^{-1}([a,M])$ is a closed subset of the compact set $\operatorname{supp} f$, so is compact. It is a G_δ because $f^{-1}([a,\infty)) = \bigcap_{n\geq 1} \underbrace{f^{-1}((a-\tfrac{1}{n},\infty))}_{=\text{open (f cont.!)}}$.

(b) By hypothesis, $K = \bigcap_{n\geq 1} U_n$. Use Urysohn's Lemma to find $\chi_K \leq f_n \prec U_n$, and let $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. Since $0 \leq f_n \leq 1$, this converges uniformly, so f is continuous. We may also assume that all U_n are contained in a compact set L (since X = LCH; see Prop. 4.31). Then $f_n = 0$ on L^c, so $f \in C_c(X)$. Clearly $0 \leq f \leq 1$, and $f = 1$ on K since $f_n = 1$ on K for all $n \geq 1$. If $x \notin K$, then $x \notin U_n$ for some n, so $f_n(x) = 0$ and $f(x) \leq 1 - 2^{-n} < 1$. Thus $K = f^{-1}(\{1\})$, as requested.

(c) Recall $\mathcal{B}^0 :=$ σ-alg. generated by $f^{-1}([a,\infty))$, $a>0$, $f \in C_c(X)$. Write $\mathcal{G} :=$ σ-alg. generated by the compact G_δ's. We want to show that $\mathcal{B}^0 = \mathcal{G}$: "$\subset$" follows from (a), and "$\supset$" follows from (b) (recall also that $f^{-1}(\{1\}) = \bigcap_{n\geq 1} \underbrace{f^{-1}([1-1/n, 1+1/n))}_{=[1-1/n,\infty)\setminus[1+1/n,\infty)}$ is in $\mathcal{B}^0$.

7.1.5: (a) Let $U_1, U_2, \ldots$ be a basis of $\mathcal{T}$ (= every open $V \subset X$ is a union of U_j's). For each $x \notin K$, we can find a neighborhood V_x s.t. $\overline{V_x} \cap K = \emptyset$ (Hd property; see Prop. 4.30 for the extra claim that the closure of V_x can be taken disjoint from K). Now pick an $n = n(x)$ with $U_n \subset V_x$ (so $\overline{U_n} \subset \overline{V_x}$ and thus $\overline{U_n} \cap K = \emptyset$ also).
Then $K = \bigcap_{\substack{n=n(x)\\ \text{for some } x\notin K}} \overline{U_n}^c$ (why?).

(b) In part (a), I did not use the compactness of K; I can also do this for a closed set F, and we can insist that the $\overline{U_n}$ are compact (by LC). So if F is closed, then $F = \bigcap_{\text{some } n} \overline{U_n}^c \in \mathcal{B}^0$, by (a), since $\overline{U_n} \in \mathcal{B}^0$. } "cpt. ⇝ cpt. G_δ"

So $\mathcal{B}^0 \supset \mathcal{B}$, and $\mathcal{B}^0 \subset \mathcal{B}$ is trivial.