(1) (a) $A \mapsto \int_A X\,dP$ defines a signed measure on $(\Omega, \mathcal{A})$, } use DC to verify σ-additivity
$\hat{\mathcal{A}}$ and clearly $\int_A X\,dP = 0$ if $P(A)=0$, so Radon – Nikodym applies and gives an f^A as desired. Uniqueness is also part of R-N (=Thm 3.8).

(b) Clear; see HW #3, 1(b).

(c) Write $A_c := Y^{-1}(\{c\})$. If we had $x_1, x_2 \in f(A_c)$, $x_1 \neq x_2$, then $f^{-1}(\{x_1\}) \cap A_c \neq \emptyset, A_c$ and thus $f^{-1}(\{x_1\})$ is not of the form $Y^{-1}(B)$ for any $B \subset \mathbb{R}$ (try $c \in B$ and $c \notin B$!) $\rightsquigarrow$ f not $\mathcal{A}_Y$-measurable.
Now let $\varphi(c) = f(x)$, $x \in A_c$ (and $\varphi(d) = 0$, say, if $d \notin Y(\Omega)$). Then $f = \varphi \circ Y$.

(d) Y takes the values $0, 1, 2$ on $\{(00)\}$, $\{(01), (10)\}$, $\{(11)\}$, respectively, so $\mathcal{A}_Y$ is the σ-algebra generated by these sets.
By (c), $E(X|Y) \equiv f$ is constant on these sets, and from (1):
$f(00) \cdot \frac{1}{4} = \frac{1}{4} \cdot 0 = 0$ $\quad (A = \{(00)\})$, $\frac{1}{2} f(01) = \frac{1}{4}(0+1) = \frac{1}{4} \rightsquigarrow f(01) = f(10) = \frac{1}{2}$,
$f(11) \cdot \frac{1}{4} = \frac{1}{4} \cdot 1 \rightsquigarrow f(11) = 1$. So $E(X|Y) = \frac{1}{2} Y$.

(e) For a rectangle $B_2 = B \times B'$, we have that $P_{X,Y}(B_2) = P(X \in B, Y \in B')$ and $P_X \otimes P_Y(B_2) = P_X(B) P_Y(B')$ (and product measure $P_X \otimes P_Y$ is the only measure satisfying this). This gives the first claim.

$$\int_{\mathbb{R}\times\mathbb{R}} |xy|\, d(P_X \otimes P_Y)(x,y) \underset{\text{Tonelli}}{=} \int dP_X(x)|x| \int dP_Y(y)|y| < \infty, \text{ so } XY \in L^1(\mathbb{R}^2, P_X \otimes P_Y)$$

and (Fubini!) $EXY = \int_\Omega XY\,dP(\omega) \underset{\text{substitution rule}}{=} \int_{\mathbb{R}^2} xy\, dP_{X,Y}(x,y) = \int_{\mathbb{R}} dP_X\, x \int_{\mathbb{R}} dP_Y\, y$
$= EX \cdot EY$

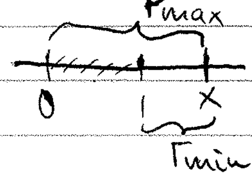
(f) Claim: X, Y independent $\Rightarrow E(X|Y) = EX$
This is clearly $\mathcal{A}_Y$-measurable (constant function!), and if $A \in \mathcal{A}_Y$, $A = Y^{-1}(B)$, then X, χ_A are also independent because $\chi_A(\omega) = \begin{cases} 1 \Leftrightarrow Y(\omega) \in B \\ 0 \Leftrightarrow Y(\omega) \in B^c \end{cases}$.
Now (e) shows that $\int_A E(X)\,dP = EX \cdot P(A)$
and $\int_A X\,dP = \int_\Omega X\chi_A\,dP \overset{\text{(e)}}{=} EX \cdot E\chi_A = EX \cdot P(A)$ " ✓

$E(X|X) = X$, since X is (obviously) $\mathcal{A}_X$-measurable and (1) holds trivially.

6.5.43: (i) $0 < x < 1$: $(Hf)(x) = 1$

} just take $r > 0$ small to see that $(Hf)(x) \geq 1$; $Hf \leq 1$ is clear since $|f| \leq 1$

(ii) $x > 1$:



It suffices to consider $x - 1 \leq r \leq x$ (otherwise $\rightarrow$ $\int_{x-r}^{x+r} |f| = 0$).

In this case, $\int_{x-r}^{x+r}|f|\,dt = \int_{x-r}^{1} dt = 1+r-x$, so $\frac{1}{2r}\int\cdots = \frac{1}{2} + \frac{1-x}{2r}$,
which is maximized by taking $r=x$. Thus $(Hf)(x) = \frac{1}{2x}$.

(iii) $x<0$: $(Hf)(x) = \frac{1}{2[\mathrm{dist}(x,(0,1))+1]} = \frac{1}{2(1+|x|)}$ by symmetry.

Since $Hf \le 1$, $(Hf)(x) \sim c/|x|$ for large $|x|$, it is clear that $f \in L^p$ for $p>1$ (this also follows from Corollary 6.35) and $f \notin L^1$.

If $0<\alpha<1$, then $Hf>\alpha$ on $0 \le x \le 1$ and $1 \le x < \frac{1}{2\alpha}$ (if $\alpha<\frac{1}{2}$) and $1-\frac{1}{2\alpha} < x \le 0$ ($\alpha < \frac{1}{2}$), so $m(Hf>\alpha) = \frac{1}{\alpha}-1$ ($\alpha<\frac{1}{2}$).
Since $m(Hf>\alpha)=0$ for $\alpha \ge 1$, this shows that $Hf \in$ weak L^1.

Finally, $\|Hf\|_p^p = \int_{-\infty}^{0} \frac{dx}{2^p(1+|x|)^p} + 1 + \int_1^\infty \frac{dx}{2^p x^p} = 1 + 2^{1-p}\int_1^\infty \frac{dx}{x^p}$

$$= \underbrace{1+\frac{2^{1-p}}{-1+p}}_{\xrightarrow[p\to 1+]{} 1}, \text{ so } (p-1)\|Hf\|_p = (p-1)\left(1+\frac{2^{1-p}}{p-1}\right)^{1/p} = \left((p-1)^p + 2^{1-p}\underbrace{(p-1)^{p-1}}_{\xrightarrow[p\to 1+]{} 1}\right)^{1/p}$$

$\}\ x^x = e^{x\ln x} \xrightarrow[x\to 0+]{} 1$