(1.) (a) First of all, $F^{-1}(E) \in \mathcal{M}$ if $E \in W$ (why?), so $\nu(E)$ is well defined.
$\nu(\phi) = 0$, and if $E_j \in W$ are disjoint, $E := \bigcup_{j\geq 1} E_j$, then $F^{-1}(E) = \dot{\bigcup}_{j\geq 1} F^{-1}(E_j)$,
so $\nu(E) = \sum_{j\geq 1} \mu(F^{-1}(E_j)) = \sum_{j\geq 1} \nu(E_j)$.

If $(a,b] \subset (0,\infty)$, then $-\lambda_f((a,b]) = \lambda_f(a) - \lambda_f(b) = \mu(\{x : |f(x)| > a\})$
$- \mu(\{x : |f(x)| > b\}) = \mu(\{x : a < |f(x)| \leq b\}) = \mu(F^{-1}((a,b])) = \nu((a,b])$.
So $-\lambda_f$ and ν agree on the (half-open) intervals $(a,b]$; this (as usual) implies that they are equal (on $\mathcal{B}$).
"Essentially" reminds us that in Section 6.4, $-\lambda_f$ was defined on $(0,\infty)$ (but it could have been defined on $[0,\infty)$ or $\mathbb{R}$, in the same way).

(b) If $E \in W$, then $F^{-1}(E^c) = (\underbrace{F^{-1}(E)}_{\in \mathcal{M}})^c \in \mathcal{M}$, so $E^c \in W$ also. Similarly, if $E_j \in W$ ($j \in \mathbb{N}$), then $F^{-1}(\bigcup_{j\geq 1} E_j) = \bigcup_{j\geq 1} \underbrace{F^{-1}(E_j)}_{\in \mathcal{M}} \in \mathcal{M}$, so $\bigcup_{j\geq 1} E_j \in W$.

If $F = \chi_E$, $E \notin \mathcal{M}$, then $W = \{\phi, Y\}$, so ν contains no information about F.

<u>6.3.34:</u> <u>$p>2$:</u> Since $f \in AC[\varepsilon, 1]$, $f(x) = f(1) - \int_x^1 f'(t)\,dt$ for all $x > 0$, so if we can show that $f' \in L^1(0,1)$, DC will imply that $\lim_{x\to 0+} f(x)$ exists (please make sure you understand <u>exactly</u> how this works).
By Hölder, $\int_0^1 |f'(x)|\,dx = \int_0^1 x^{1/p} |f'(x)| \cdot x^{-1/p}\,dx \leq \left(\int_0^1 x |f'(x)|^p dx\right)^{1/p} \cdot$
$\cdot \left(\int_0^1 x^{-q/p}\,dx\right)^{1/q}$ $\quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

Since $p > 2$, $q < 2$, so $q/p < 1$ and $x^{-q/p} \in L^1(0,1)$, so the above estimate does show that $f' \in L^1(0,1)$.

<u>$p=2$:</u> If $0 < x < x_0 \leq 1$, then $|f(x)| = \left| f(x_0) - \int_x^{x_0} f'(t)\,dt \right| \leq |f(x_0)| + \int_x^{x_0} |f'|\,dt$
$\underset{\text{Hölder, } p=q=2}{\leq} |f(x_0)| + \left(\int_x^{x_0} t |f'(t)|^2 dt\right)^{1/2} \left(\int_x^{x_0} t^{-2/2}\,dt\right)^{1/2} \leq |f(x_0)| + \underbrace{\left(\int_0^{x_0} t |f'(t)|^2 dt\right)^{1/2} \cdot |\ln x|^{1/2}}$
$\equiv C(x_0) + I(x_0) \cdot |\ln x|^{1/2}$ (this shows that $\frac{|f(x)|}{|\ln x|^{1/2}}$ remains bounded)
If $\varepsilon > 0$ is given, take $x_0 > 0$ so small that $I(x_0) < \frac{\varepsilon}{2}$ (possible because $I(x_0) \to 0$ as $x_0 \to 0+$, by DC). Then
$\frac{|f(x)|}{|\ln x|^{1/2}} \leq \frac{C(x_0)}{|\ln x|^{1/2}} + \frac{\varepsilon}{2}$ $\quad (x < x_0)$. This will be $< \varepsilon$ for small enough $x > 0$, since $|\ln x| \to \infty$.

$p<2$: is similar (to $p=2$):

$$|f(x)| \le |f(x_0)| + \left(\int_x^{x_0} t|f'(t)|^p dt\right)^{1/p} \left(\int_x^{x_0} t^{-q/p} dt\right)^{1/q} \quad ; \quad \frac{1}{p}+\frac{1}{q}=1$$

$$\left(\int_x^{x_0} t^{-q/p} dt\right)^{1/q} = \left(\frac{1}{\underbrace{1-q/p}_{<0}}\left(x_0^{1-q/p} - x^{1-q/p}\right)\right)^{1/q} \le C x^{1/q - 1/p} = C x^{1-2/p}$$

Thus $|f(x)| \le C(x_0) + C I(x_0) x^{1-2/p}$ $(x<x_0)$, and $I(x_0) \to 0$ $(x_0 \to 0)$.
We can now argue as above to finish the proof.

6.4.36: (a) $\lambda_f(\alpha) \le \mu(\{x : f(x) \ne 0\}) \equiv C_1 < \infty$ for all $\alpha \ge 0$, and $\alpha^p \lambda_f(\alpha) \le C_2$ $(\alpha \ge 0)$.

Thus $\int_0^\infty \alpha^{q-1}\lambda_f(\alpha) d\alpha = \int_0^1 \cdots + \int_1^\infty \cdots \le C_1 \underbrace{\int_0^1 \alpha^{q-1} d\alpha}_{<\infty \ (q>0)} + C_2 \underbrace{\int_1^\infty \alpha^{\overbrace{q-p-1}^{<-1}} d\alpha}_{<\infty} < \infty$,

so $\|f\|_q < \infty$ by Proposition 6.24.

(b) If $f \in L^\infty$, say $|f| \le C$ a.e., then $\lambda_f(\alpha) = 0$ for $\alpha \ge C$. Thus

$$\int_0^\infty \alpha^{q-1}\lambda_f(\alpha) d\alpha = \int_0^C \alpha^{q-1}\lambda_f(\alpha) d\alpha \le C_2 \int_0^C \alpha^{\overbrace{q-p-1}^{>-1}} d\alpha < \infty, \text{ so } f \in L^q.$$

6.4.38: $$\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha \overset{(MC)}{=} \underbrace{\sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \alpha^{p-1}\lambda_f(\alpha) d\alpha}_{\circledast} \ge \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \lambda_f(2^{k+1}) \underbrace{(2^{k+1}-2^k)}_{=2^k}$$

(λ_f decreasing, $p-1 \ge 0$)

$$= \sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^{k+1}) = 2^{-p}\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k), \text{ and}$$

$$\circledast \le \sum_{k=-\infty}^{\infty} 2^{(k+1)(p-1)}\lambda_f(2^k)(2^{k+1}-2^k) = \sum_{k=-\infty}^{\infty} 2^{kp+p-1}\lambda_f(2^k) = 2^{p-1}\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k)$$

This combined with Proposition 6.24 says that $\|f\|_p < \infty \iff \sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$.