(1.) $f_1 \in L^\infty$, $f_1 \notin L^p$ for $p<\infty$;
$f_2 \notin L^p$ for all p;
$f_3(x)=0$ a.e., so $f_3 \in L^p$ for all p

$\int |f_4|^p dx = \int_1^\infty x^{-p} dx < \infty \Leftrightarrow p>1$. Since also $f_4 \in L^\infty$, we have that $f_4 \in L^p$ for $p>1$, $f_4 \notin L^1$

$f_5 \notin L^\infty$, and $\int |f_5|^p dx = \int_0^1 x^{-p/2} dx < \infty \Leftrightarrow \frac{p}{2}<1$ or $p<2$, so $f_5 \in L^p$ for $1 \le p < 2$, $f_5 \notin L^p$ for $p \geq 2$

$\int |f_6|^p dx = \int_0^\infty x^{-p} dx = \infty$ for all $p \geq 1$ ($p>0$, in fact) because the integral is divergent either near $x=0$ or at infinity, so $f \notin L^p$ for all $p \geq 1$

(2.) (a) $\mu(E)=0 \Rightarrow E=\phi$, thus $\operatorname{ess\,sup} |a_n| = \sup |a_n|$.

(b) By the homogeneity of the norms $\|\cdot\|_p$, it suffices to do this for $\|a\|_\infty = 1$. Since $a \in \ell_1$, it follows that $a_n \to 0$, and $\sup|a_n| = \max |a_n|$. So $|a_{n_0}|=1$ for some $n_0 \geq 1$, and thus $\|a\|_p = \left(\sum_{n=1}^\infty |a_n|^p\right)^{1/p} \geq (|a_{n_0}|^p)^{1/p} = 1$.
On the other hand, since $|a_n| \le \|a\|_\infty = 1$, $\left(\sum_{n=1}^\infty |a_n|^p\right)^{1/p} \le \|a\|_1^{1/p} \xrightarrow[p\to\infty]{} 1$.

6.1.5: $p<q$: $L^p \not\subset L^q \Leftrightarrow$ For every $\varepsilon>0$, there exists $E \in \mathcal{M}$ with $0<\mu(E)<\varepsilon$.

"$\Rightarrow$": Suppose there exists $\varepsilon>0$ so that $\mu(E) \geq \varepsilon$ if $\mu(E)>0$, and let $f \in L^p$. Define $E_n = \{x \in X : 2^n < |f(x)| \le 2^{n+1}\}$ ($n \in \mathbb{Z}$).
Then $\sum_{n=-\infty}^{\infty} \mu(E_n) 2^{np} \le \int |f|^p < \infty$ and

$$\int |f|^q \underset{MC}{=} \sum_{n=-\infty}^{\infty} \int_{E_n} |f|^q \le \sum_{n=-\infty}^{\infty} \mu(E_n) 2^{(n+1)q} = 2^q \sum \left(\mu(E_n) 2^{np}\right)^{q/p} \frac{1}{\mu(E_n)^{\underbrace{q/p - 1}_{>0}}}$$

$$\le \frac{2^q}{\varepsilon^{q/p-1}} \sum \left(\mu(E_n)2^{np}\right)^{q/p} < \infty \text{ because } q/p > 1,\ \sum \mu(E_n) 2^{np} < \infty.$$

Thus $f \in L^q$ and $L^p \subset L^q$.

"$\Leftarrow$": Pick $F_n \in \mathcal{M}$ with
$0 < \mu(F_1) < 2^{-1}$, $0<\mu(F_2) < \min\{2^{-2}, \frac{1}{4}\mu(F_1)\}$,
$0<\mu(F_3) < \min\{2^{-3}, \frac{1}{4^2}\mu(F_1), \frac{1}{4}\mu(F_2)\}$ etc., and put

$\longrightarrow$

$E_1 = F_1 \setminus \bigcup_{j\geq 2} F_j$, $E_2 = F_2 \setminus \bigcup_{j\geq 3} F_j$ etc.

Then indeed $0 < \mu(E_n) < 2^{-n}$, $E_m \cap E_n = \phi$ $(m \neq n)$.

Let $f = \sum_{n=1}^{\infty} n^{-2/p} \mu(E_n)^{-1/p} \chi_{E_n}$. Then $\int |f|^p d\mu = \sum_{n\geq 1} n^{-2} \mu(E_n)^{-1} \mu(E_n)$ $= \sum_{n\geq 1} \frac{1}{n^2} < \infty$, but

$\int |f|^q d\mu = \sum_{n\geq 1} n^{-2q/p} \mu(E_n)^{1-q/p} = \infty$, because

$n^{-2q/p} \mu(E_n)^{\underbrace{1-q/q}_{<0}} \geq n^{-2q/p} 2^{n(q/p - 1)} \to \infty$.

$L^q \not\subset L^p \iff$ For every $C>0$, there exists $E \in \mathcal{M}$ with $C < \mu(E) < \infty$.

"$\Rightarrow$": Suppose that $\mu(E) \leq C$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$ (for some $C>0$).
Let $f \in L^q$. Since $\{x \in X : |f(x)| > 0\} = \bigcup_{n\geq 1} \underbrace{\{x : |f(x)| \geq 1/n\}}_{=:\,E_n}$, $\mu(E_n) < \infty$,
thus $\mu(E_n) \leq C$ $= \mu(\bigcup_{n=1}^{N} E_n) < \infty$, thus $\leq C$, we obtain from MC (or Thm 1.8(c))
that also $\mu(\underbrace{\{x : |f(x)| > 0\}}_{=:\,E}) \leq C$. Since $\|f\|_p = \|\chi_E f\|_p$, we are now in
the situation of Proposition 6.12, and it follows that $f \in L^p$ also.

"$\Leftarrow$": Construct again F_n's (inductively) with $1 \leq \mu(F_1) < \infty$, $1 + \mu(F_1) \leq \mu(F_2) < \infty$, $1 + \mu(F_1) + \mu(F_2) \leq \mu(F_3) < \infty$ etc., and let $E_1 = F_1$, $E_2 = F_2 \setminus F_1$, $E_3 = F_3 \setminus (F_1 \cup F_2)$ etc. This gives the E_n's from the hint.

If now $f = \sum n^{-(1+\varepsilon)/q} \mu(E_n)^{-1/q} \chi_{E_n}$, then $\int |f|^q d\mu = \sum n^{-1-\varepsilon} < \infty$,
but $\int |f|^p d\mu = \sum n^{-(1+\varepsilon)p/q} \underbrace{\mu(E_n)}_{\geq 1}{}^{\overbrace{1-p/q}^{>0}} \geq \sum n^{-(1+\varepsilon)p/q} = \infty$ for
small $\varepsilon > 0$ ($p/q < 1$, so $(1+\varepsilon)p/q < 1$ for small $\varepsilon > 0$).

The first part also works for $q = \infty$ (with adjusted proofs).
In the second part, "$\Rightarrow$" fails: $X = \{a, b\}$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(\{a\}) = 1$, $\mu(\{b\}) = \infty$
(this measure is <u>not</u> σ-finite!): Then $\mu(E) \leq 1$ if $\mu(E) < \infty$, but $L^\infty \not\subset L^p$
if $p < \infty$: $f(a) = f(b) = 1$ $\in L^\infty$, but $f \notin L^p$.

<u>6.1.11:</u> (a) If $z \notin R_f$, then $\mu(\{x : |f(x) - z| < \varepsilon\}) = 0$ for some $\varepsilon > 0$.
But then also $\mu(\{x : |f(x) - w| < \varepsilon/2\}) = 0$ for all $w \in \mathbb{C}$ with $|w - z| < \varepsilon/2$,
so $B(\varepsilon/2, z) \subset R_f^c$ and R_f^c is open. $\longrightarrow$

(b) R_f must be bounded if $f \in L^\infty$, so it's compact by (a).
From the definition of R_f, we see that $|f(x)| \geq |z| - \varepsilon$ on a positive measure set for all $\varepsilon > 0$ and all $z \in R_f$, thus $\|f\|_\infty \geq \max R_f$.
On the other hand, if K is a compact subset of R_f^c, then, for every $z \in K$, we can find a ball $B(r_z, z)$ so that $\mu(\{x : f(x) \in B(r_z, z)\}) = 0$. Since finitely many of these balls suffice to cover K, it follows that $\mu(\{x : f(x) \in K\}) = 0$. Since R_f^c can be written as a countable union of such sets K_n, it follows that $\mu(\{x : f(x) \notin R_f\}) = 0$. In other words, $f(x) \in R_f$ a.e. and thus $\|f\|_\infty \leq \max R_f$.