5483 HW #1 (1)

(27) (a) $\sum_{j=1}^{N} |F(x_j)-F(x_{j-1})| = F(x)-F(x_0) \le F(x)-F(-\infty)$ and $\to F(x)-F(-\infty)$ as $x_0 \to -\infty$. Thus $T_F(x) = F(x)-F(-\infty)$.

(b) $\sum_{j=1}^{N} |(aF+bG)(x_j)-(aF+bG)(x_{j-1})| \le |a|\sum_{j=1}^{N}|F(x_j)-F(x_{j-1})| + |b|\sum_{j=1}^{N}|G(x_j)-G(x_{j-1})|$
$\le |a|T_F(x) + |b|T_G(x)$

(c) $\sum_{j=1}^{N}|F(x_j)-F(x_{j-1})| = \sum_{j=1}^{N}|F'(\xi_j)|(x_j-x_{j-1}) \le M(b-a)$ $\{ |F'| \le M$

(d) $F \in BV[a,b]$ by (c). If $x_j = \frac{\pi}{2}+j\pi$, then $\sum_{j=1}^{N}|\sin x_j - \sin x_{j-1}| = 2N$, and this becomes arbitrarily large as $N \to \infty$.

(e) let's say $a \le 0 < b$. Let $\frac{1}{x_j} = \frac{\pi}{2}+(N-j)\pi$, $j=0,1,\dots,N$. Then $|F(x_j)-F(x_{j-1})| \ge 2x_{j-1}$, so $\sum_{j=1}^{N}|F(x_j)-F(x_{j-1})| \ge 2\sum_{k=1}^{N}\frac{1}{\pi/2+k\pi} \ge C\sum_{k=1}^{N}\frac{1}{k}$. Since $\sum\frac{1}{k}$ diverges, this can be made arbitrarily large.
$\{$ Notice that ~~$T_F(b) < T_F(\delta) < \infty$~~ $T_F([\delta,b])<\infty$ for all $\delta>0$, so there's no problem here if $x_j > b$ for $j = M,\dots,N$.

(30.) $\mathbb{Q} = \{q_n : n \ge 1\}$, $F(x) = \sum_{n:\, q_n \le x} 2^{-n}$ $\{$ this is the function associated with $\mu = \sum 2^{-n}\delta_{q_n}$ $\rightsquigarrow$ discontinuous exactly at $x \in \mathbb{Q}$ (this can be proved directly)

(31.) (a) This is obvious at $x \ne 0$.
$\frac{F(h)-F(0)}{h} = h\sin h^{-1} \underset{h\to 0}{\to} 0 \rightsquigarrow F'(0)$ exists and $=0$.
Similarly, $G'(0)=0$.

(b) $F'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \ne 0 \\ 0 & x=0 \end{cases}$ is bounded on $-1 \le x \le 1$, so $F \in BV[-1,1]$ by 27(c).

As for G, we proceed as in 27(e): Let $\frac{1}{x_j^2} = \frac{\pi}{2}+(N-j)\pi$, $j=0,1,\dots,N$. Then $|G(x_j)-G(x_{j-1})| \ge 2x_{j-1}^2$. As above, the sum of these terms is $\gtrsim \sum_{j=1}^{N}\frac{1}{j}$, so can be made large.

(39.) As in the hint, I'll assume F, F_j right-continuous and $F_j(a)=F(a)=0$. Compare Thm 3.23(b) for this. Let $\mu(B) = \sum \mu_{F_j}(B)$; this defines a finite measure on $[a,b]$ (MC or Tonelli for σ-additivity). Notice that $\mu = \mu_F$, by comparing these for $B=[a,x]$. I claim that $\mu = (\sum \mu_{j,ac}) + (\sum \mu_{j,s})$ is the Lebesgue decomposition of μ:
(i) If $m(B)=0$, then $\mu_{j,ac}(B)=0$, so $(\sum\mu_{j,ac})(B)=0$, so $\sum\mu_{j,ac} \ll m$;
(ii) $\mu_{j,s}$ is supported by S_j, $m(S_j)=0 \rightsquigarrow \sum\mu_{j,s}$ supported by $\cup S_j$
Thus $\mu_{ac}(B) = \int_B F'(x)dx = \sum\mu_{j,ac}(B) = \sum\int_B F_j'(x)dx \underset{MC}{=} \int_B(\sum_j F_j'(x))dx \rightsquigarrow F' = \sum F_j'$ a.e.

(40.) (1) G strictly increasing: If $x<y$, then $x < a_n < b_n < y$ for some $a_n < b_n \in \mathbb{Q}$, so $G(y)-G(x) > 2^{-n}$. $\longrightarrow$

(2) G continuous: G is the uniform limit of the continuous functions $\sum_{n=1}^{N} 2^{-n} F_n$, as $N \to \infty$.

(3) $G' = 0$ a.e.: By 39, $G' \underset{a.e.}{=} \sum_{n=1}^{\infty} 2^{-n} F_n' \underset{a.e.}{=} 0$.