Homework # 9: 5463: Real Analysis 2

1. Do the following maps on test functions define distributions?

$$\langle F, \varphi \rangle = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} 2^{-n},$$
$$\langle G, \varphi \rangle = \sum_{n=0}^{\infty} 2^n \varphi^{(n)}(n),$$
$$\langle H, \varphi \rangle = \sum_{n=0}^{\infty} (\varphi(n))^n.$$

2. Show that $n^k e^{inx} \to 0$ as $n \to \infty$ in $\mathcal{D}'(\mathbb{R})$, for any (integer) $k \geq 0$.

   Suggestion: Show the general result that if $F_n \to F$ in $\mathcal{D}'$, then also $F_n' \to F'$.

3. *(alternative definition of the distributional derivative)* (a) Show that if $\varphi \in C_c^\infty(\mathbb{R})$ is a test function, then

$$\frac{\varphi(x+h) - \varphi(x)}{h} \to \varphi'(x) \qquad (h \to 0),$$

   uniformly on $x \in \mathbb{R}$. (The point is the uniform convergence, otherwise this would just be the definition of φ'.)
   (b) Use part (a) to show that for any $F \in \mathcal{D}'(\mathbb{R})$,

$$\frac{1}{h} (\tau_{-h} F - F) \to F'$$

   in $\mathcal{D}'$ (in particular, the limit on the left-hand side always exists).

4. Let $(X, \mathcal{T})$ be a topological space, and let $Y \in \mathcal{B}_X$ be a Borel subset. Then Y is a topological space itself, with the induced topology (so what are the open sets?), and thus has a Borel σ-algebra $\mathcal{B}_Y$. Show that $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}_X\}$.

“due” 5/1