Homework # 5: 5463: Real Analysis 2

1. Let X_0 be an LCH space, and let $X = X_0 \cup \{\infty\}$ be its *one-point compactification.*

   (a) Make sure you're fully familiar with the definition and basic properties of X (what are the open sets?). See pg. 132 for a quick description.

   (b) $C_c(X_0) \subset C(X)$, in the following sense: If $f_0 \in C_c(X_0)$, then $f \in C(X)$, where
   $$f(x) = \begin{cases} f_0(x) & x \in X_0 \\ 0 & x = \infty \end{cases}.$$

   (c) So, if I is a positive linear functional on $C(X)$, we also obtain a positive linear functional I_0 on $C_c(X)$ by restricting (more precisely, we define $I_0(f_0) := I(f)$, where f is related to f_0 as in (b)). What is the relation between the corresponding Radon measures μ_0, μ? How can you obtain μ_0 from μ?

   (d) Prove that, conversely, it is not necessarily true that every positive linear functional I_0 on $C_c(X_0)$ has an extension to a positive linear functional I on $C(X)$.

   (e) However, if I_0 is a positive linear functional on $C_0(X_0)$ (see pg. 132 again for the definition and other useful information), then there will be positive linear extensions to $C(X)$. Show this; can you in fact describe all such extensions and the relation between the corresponding measures μ_0 and μ?

2. Exercises 3, 4, 5 from Section 7.1

"due" 3/6