Homework # 4: 5463: Real Analysis 2

1. Let $(\Omega, \mathcal{M}, P)$ be a probability space (= measure space with $P(\Omega) = 1$), let $X \in L^1(\Omega)$ be a (real valued) random variable, and let $\mathcal{A} \subset \mathcal{M}$ be a σ-subalgebra of $\mathcal{M}$.

   (a) Prove that there exists an $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$-measurable function $f : \Omega \to \mathbb{R}$ so that
   $$\int_A f(\omega)\, dP(\omega) = \int_A X(\omega)\, dP(\omega) \qquad \text{for all } A \in \mathcal{A}. \tag{1}$$
   Moreover, if g is another such function, then $f = g$ almost surely (= P-almost everywhere).
   This function is usually denoted by $E(X|\mathcal{A})$ and called the *conditional expectation* of X, given $\mathcal{A}$.
   Hint: The right-hand side of (1) defines a (signed) measure on $(\Omega, \mathcal{A})$. Use the Radon-Nikodym Theorem.

   (b) If $Y : \Omega \to \mathbb{R}$ is another random variable, then
   $$\mathcal{A}_Y = \left\{Y^{-1}(B) : B \in \mathcal{B}_{\mathbb{R}}\right\}$$
   defines a σ-algebra on Ω, and $\mathcal{A}_Y \subset \mathcal{M}$. We can therefore define the *conditional expectation of X, given Y* by
   $$E(X|Y) = E(X|\mathcal{A}_Y).$$

   (c) Prove that if $f : \Omega \to \mathbb{R}$ is $\mathcal{A}_Y$-measurable, then f is constant on every set of the form $Y^{-1}(\{c\})$. Then show that this implies that f can be written as a function of Y: $f = \varphi \circ Y$ for some $\varphi : \mathbb{R} \to \mathbb{R}$. (With more effort, one can show that φ can be chosen as a Borel function.)
   So $E(X|Y)$ can be thought of as that function of Y that produces the correct partial averages of X.

   (d) Find $\mathcal{A}_Y$ and $E(X|Y)$ in the following example: $\Omega = \{(00), (01), (10), (11)\}$, $\mathcal{M} = \mathcal{P}(\Omega)$, $P(\{\omega\}) = 1/4$ for all $\omega \in \Omega$, $X((ab)) = a$, $Y((ab)) = a + b$

   (e) Random variables X, Y are called *independent* if
   $$P(X \in B, Y \in B') = P(X \in B)P(Y \in B') \qquad \text{for all } B, B' \in \mathcal{B}_{\mathbb{R}}.$$
   Show that X, Y are independent if and only if the *joint distribution* $P_{X,Y}$ is the product (in the sense of Section 2.5) $P_X \otimes P_Y$. Here,
   $$P_{X,Y}(B_2) = P((X,Y) \in B_2) \quad (B_2 \in \mathcal{B}_{\mathbb{R}^2}), \qquad P_X(B) = P(X \in B) \quad (B \in \mathcal{B}_{\mathbb{R}}).$$
   (See also pg. 316.) Then use Fubini-Tonelli to show that if $X, Y \in L^1(\Omega)$ are independent, then
   $$E(XY) = E(X)E(Y) \qquad \left(E(Z) := \int_\Omega Z(\omega)\, dP(\omega)\right).$$

   (f) What is $E(X|Y)$ if X, Y are independent? What is $E(X|X)$?

2. Exercise 43 from Section 6.5

Discussion: 2/14