Homework # 3: 5463: Real Analysis 2

1. Let μ be a measure on $(X, \mathcal{M})$, and let $(Y, \mathcal{N})$ be another space with a σ-algebra ("measurable space"). Let $F : X \to Y$ be a measurable function.

   (a) Prove that
   $$\nu(E) = \mu(F^{-1}(E)) \qquad (E \in \mathcal{N})$$
   defines a new measure on the range Y of F (the *distribution* of F). Show also that if $f : X \to \mathbb{C}$, $F = |f|$, so $Y = [0, \infty)$, then ν essentially agrees with the measure $-\lambda_f$ from Section 6.4 (why only "essentially"?).

   (b) Even if no σ-algebra on Y is given and $F : X \to Y$ is an arbitrary map, show that we can still run the same construction, as follows: Let
   $$\mathcal{N} = \{E \subset Y : F^{-1}(E) \in \mathcal{M}\}.$$
   Prove that this defines a σ-algebra on Y, and (of course) F becomes $(\mathcal{M}, \mathcal{N})$-measurable.

   Probably the situation in (a) is preferable; consider for example the function $F = \chi_E$ with $E \notin \mathcal{M}$. What σ-algebra $\mathcal{N}$ on $Y = \{0, 1\}$ is obtained in this case?

2. Exercises 34, 36, 38 from Sections 6.3, 6.4

   Comments: (i) Exercise 34: No tools from Section 6.3 are needed to do this; Hölder's inequality should suffice. As the first step, try to deduce that the quantities in question are bounded (if $p \leq 2$; the case $p > 2$ is easier). Can you then improve this argument to obtain the full claim?

   (ii) Exercises 36, 38: Proposition 6.24 should be useful here. In Exercise 36, what do the assumptions $\mu(\{x : f(x) \neq 0\}) < \infty$ and $f \in L^\infty$ tell you about the function $\lambda_f(\alpha)$?

Discussion: 2/7