HARMONIC ANALYSIS ON $SO(3)$

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These notes are meant to give a glimpse into non-commutative harmonic analysis by looking at one example. I will follow Dym-McKean, Fourier Series and Integrals, Sect. 4.8 – 4.13, very closely.

1. The group $SO(3)$

Since Fourier analysis on finite abelian groups worked so well, we now get (much) more ambitious and discuss an infinite non-abelian group. Our example is the group of proper rotations on $\mathbb{R}^3$, now denoted by $SO(3)$ ("special orthogonal group" – "special" just means that the determinant is equal to 1). So

$$SO(3) = \{ g \in \mathbb{R}^{3 \times 3} : g^t g = 1, \det g = 1 \}.$$ 

Such a rotation $g$ can be described by three parameters. For instance, if you know the axis of rotation (specified by a direction or a point on $S^2$ or two angles) and the angle of rotation (one parameter), $g$ is determined uniquely. Alternatively, a matrix $g \in \mathbb{R}^{3 \times 3}$ has 9 entries, but the requirement that $g^t g = 1$ gives 6 conditions on these entries, and again $9 - 6 = 3$. (The condition that $\det g = 1$ singles out one half of the matrices satisfying $g^t g = 1$; it does not reduce the dimension.) Summarizing in fancy language and adding some precision, we have:

**Theorem 1.1.** $SO(3)$ is a (compact) 3-dimensional manifold (whatever that means).

We can’t use characters to analyze functions on $G = SO(3)$. This does not come as a surprise because $G$ is not commutative and a character $\chi$ can’t distinguish between $gh$ and $hg$:

$$\chi(gh) = \chi(g)\chi(h) = \chi(hg)$$

More to the point, it can be shown that the only character $\chi$ on $SO(3)$ is the trivial character $\chi(g) \equiv 1$.

To analyze functions on $G$, we break $G$ into smaller pieces. Let $K$ be the subgroup of rotations about the $z$ axis. Equivalently, $K$ is the set of rotations that fix the north pole $n = (0,0,1)^t$. An explicit description
of $K$ is given by

$$K = \{ k(\varphi) : 0 \leq \varphi < 2\pi \}, \quad k(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

$K$ is a subgroup of $G$, and in fact $K \cong S^1$. Indeed, the map $k(\varphi) \mapsto e^{i\varphi}$ is an isomorphism from $K$ onto $S^1$.

Exercise 1.1. Check this.

We now introduce the cosets of $K$

$$gK = \{ gk : k \in K \}$$

and the set of cosets

$$G/K = \{ gK : g \in G \}.$$ 

(Warning for readers with some knowledge of group theory: $K$ is not a normal subgroup and $G/K$ is not a group.)

Exercise 1.2. Prove that two cosets $g_1K$, $g_2K$ are either equal or disjoint.

Given $h \in G$ and a coset $gK$, the group element $h$ acts on the coset $gK$ in a natural way and produces the new coset $hgK$. The next theorem shows that the coset space $G/K$ can be naturally identified with $S^2$. Moreover, if looked at on $S^2$, the above action becomes the map $x \mapsto hx$ ($x \in S^2$, $h \in SO(3)$).

**Theorem 1.2.** There exists a bijective map $j : G/K \to S^2$ so that $j(hgK) = hj(gK)$ for all $g, h \in G$.

**Proof.** Let $n = (0,0,1)^t$ be the north pole. We would like to define $j(gK) = gn$ but before we can do this, we must check that the right-hand side is independent of the choice of the representative $g$. In other words, if $g_1K = g_2K$, then we must also have that $g_1n = g_2n$. Now if $g_1K = g_2K$, then $g_2 = g_1k$ for some $k \in K$ and since $k$ fixes the north pole, $g_2n = g_1kn = g_1n$, as desired.

It is clear that $j$ satisfies $j(hgK) = hj(gK)$. Moreover, $j$ maps $G/K$ onto $S^2$ because for every $x \in S^2$, there exists a rotation $g$ so that $gn = x$. It remains to show that $j$ is injective. If $g_1n = g_2n$, then the rotation $g_2^{-1}g_1$ fixes $n$ and thus must be in $K$. But then $g_1K = g_2K$, so $g_1, g_2$ actually represent the same coset. 

We have already seen that we can let group elements act on cosets $gK$. We will now be especially interested in the double coset space

$$K/G/K = \{ KgK : g \in G \},$$
where, as expected,

\[ KgK = \{k_1gk_2 : k_1, k_2 \in G\}. \]

Things become very transparent if we use the identification \( G/K \cong S^2 \) from above. Then \( gK \) corresponds to a point \( x \) on \( S^2 \), and \( k \in K \) acts on this by just doing the rotation \( kx \). Now \( K \) is precisely the set of rotations about the \( z \) axis, so \( KgK \cong Kx \) is a circle of constant latitude on the sphere. In particular, we can parametrize the elements of \( K/G/K \) by using this latitude \( \theta \). In other words, \( \theta \) is the angle a vector pointing towards the circle makes with the \( z \) axis, and \( 0 \leq \theta \leq \pi \).

### 2. Integration on \( G \)

We can’t make any serious progress without being able to integrate functions defined on \( G \). There is heavy machinery that addresses this issue in a very general setting, but we don’t need any of this here. We just recall from the previous section that we can naturally identify \( G \cong G/K \times K \) and also \( G/K \cong S^2 \), \( K \cong S^1 \), and we do know how to integrate on \( S^1 \) and \( S^2 \), respectively. This then automatically gives us an integral on \( G \).

To carry out this program, associate with a (sufficiently nice) function \( f : G \to \mathbb{C} \) its average \( f_0 \) over \( gK \):

\[ f_0(g) = \int_K f(gk) \, dk \]

More precisely, we actually do the integral

\[
\frac{1}{2\pi} \int_0^{2\pi} f(gk(\varphi)) \, d\varphi,
\]

making use of the existing integration theory on \( S^1 \cong [0, 2\pi) \). However, at least for theoretical use of the integral, it’s usually better to be less explicit in the notation.

The point is that \( f_0 \) only depends on the coset \( gK \) of \( g \), not on \( g \) itself. In a sense, this is clear because \( f_0 \) was defined as the average over that coset. The formal proof depends on the (left and right) invariance of the integral on \( K \): For every continuous (say) function \( f : K \to \mathbb{C} \) and \( k' \in K \),

\[
\int_K f(k) \, dk = \int_K f(k'k) \, dk = \int_K f(kk') \, dk.
\]

**Exercise 2.1.** Prove (2.1). (The proof consists of unwrapping the definitions.)
Now (2.1) indeed shows that for arbitrary $k' \in K$,
\[ f_0(gk') = \int_K f(gk'k) \, dk = \int_K f(gk) \, dk = f_0(g). \]
This says that $f_0$ is constant on every coset $gK$. In particular, making use of the identification $G/K \cong S^2$, we can define
\[ (2.2) \quad \int_G f(g) \, dg = \frac{1}{4\pi} \int_{S^2} d\sigma(x) f_0(x). \]
Again, this is actually short-hand for the more precise formula
\[ \int_G f(g) \, dg = \frac{1}{4\pi} \int_{S^2} d\sigma(x) f_0(j^{-1}(x)), \]
where $j^{-1}$ is the inverse of the identification map $j : G/K \to S^2$ from Theorem 1.2. Even this is not totally accurate, we would actually need the function $\bar{f}_0 : G/K \to \mathbb{C}$ induced by $f_0 : G \to \mathbb{C}$ in the integral. Of course, (2.2) is the version we’ll work with.

The factor $1/4\pi$ makes sure that the integral is normalized in the sense that $\int_G dg = 1$. It is also left-invariant, that is,
\[ (2.3) \quad \int_G f(hg) \, dg = \int_G f(g) \, dg. \]
In fact, $dg$ is the only measure on $SO(3)$ with these properties. It is called the Haar measure.

**Exercise 2.2.** Prove (2.3). Again, you will need to unwrap the definitions.

The Haar measure on $SO(3)$ has additional nice properties:

**Theorem 2.1.** Let $f : G \to \mathbb{C}$ a continuous (say) function and $h \in G$. Then
\[ \int_G f(g) \, dg = \int_G f(g^{-1}) \, dg = \int_G f(gh) \, dg = \int_G f(hg) \, dg. \]

**Proof.** Given $f$, define a new function $f^{-1}$ by $f^{-1}(g) = f(g^{-1})$. Left-invariance of $dg$ (see (2.3)) then shows that
\[
\begin{align*}
\int_G f(g) \, dg &= \int_G f(h^{-1}g) \, dg = \int_G dh \int_G dg f(h^{-1}g) \\
&= \int_G dg \int_G dh f^{-1}(g^{-1}h) = \int_G dg \int_G dh f^{-1}(h) \\
&= \int_G f^{-1}(h) \, dh = \int_G f(g^{-1}) \, dg.
\end{align*}
\]
Given this and left-invariance, the right-invariance now follows from the calculation
\[
\int_G f(gh) \, dg = \int_G f^{-1}(h^{-1}g^{-1}) \, dg = \int_G f^{-1}(h^{-1}g) \, dg \\
= \int_G f^{-1}(g) \, dg = \int_G f(g^{-1}) \, dg = \int_G f(g) \, dg.
\]
\[\square\]

3. Convolutions

Recall that if \( X = S^1 \) or \( X = \mathbb{R}^d \), then the Fourier transform is a linear map on the functions on \( X \). Moreover, it also respects the convolution product of functions: \((f \ast g)^\wedge = \hat{f} \hat{g}\). We will now look for similar maps on functions on \( G = SO(3) \).

To do this, we must first define a convolution for functions \( f : G \to \mathbb{C} \): the obvious try is
\[
(f_1 \ast f_2)(g) = \int_G f_1(gh^{-1})f_2(h) \, dh
\]
(as usual, if in doubt, assume that \( f_1, f_2 \) are nice smooth functions; from a structural point of view, however, it would actually be best to work with the class \( L_1(G) \) of merely integrable functions here).

**Exercise 3.1.** Prove that convolution is associative.

Unfortunately, convolution is not commutative on \( SO(3) \). We can restrict attention to functions on \( G/K \) or, equivalently, functions on \( G \) that are constant on cosets. Convolution preserves this property, as is seen from the calculation
\[
(f_1 \ast f_2)(gk) = \int_G f_1(gkh^{-1})f_2(h) \, dh = \int_G f_1(gh^{-1})f_2(hk) \, dh \\
= \int_G f_1(gh^{-1})f_2(h) \, dh = (f_1 \ast f_2)(g).
\]
Here, the second equality follows from the substitution \( h \to hk \) (right-invariance!), and in the third equality, we have used the fact that \( f_2 \) is constant on the coset \( hK \).

We can go one step further and consider functions on \( K/G/K \), or, equivalently, functions on \( G \) that are constant on double cosets \( KgK \).

**Exercise 3.2.** Show that convolution preserves this property, too.

**Exercise 3.3.** Prove that \( g \) and \( g^{-1} \) have the same double coset: \( KgK = Kg^{-1}K \). In particular, \( f(g) = f(g^{-1}) \) for any function \( f \) that is constant on double cosets.
Hint: Use the representation of double cosets as circles of constant latitude on the sphere $S^2$ and observe that $\cos \theta = n \cdot gn$.

**Theorem 3.1.** If $f_1$, $f_2$ are functions on $K/G/K$, then $f_1 * f_2 = f_2 * f_1$.

**Proof.** By Exercise 3.3 and Theorem 2.1,

$$
(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h) \, dh = \int_G f_1(hg^{-1})f_2(h) \, dh
$$

$$
= \int_G f_1(h)f_2(hg) \, dh = \int_G f_2(g^{-1}h^{-1})f_1(h) \, dh
$$

$$
= (f_2 * f_1)(g^{-1}) = (f_2 * f_1)(g).
$$

□

4. Algebra homomorphisms on $L_1(K/G/K)$

Encouraged by Theorem 3.1, we now look for algebra homomorphisms $\psi : L_1(K/G/K) \to \mathbb{C}$. This is to say, we look for maps $\psi$ acting on (integrable) functions on double cosets that are linear ($\psi(af_1 + bf_2) = a\psi(f_1) + b\psi(f_2)$) and also satisfy $\psi(f_1 * f_2) = \psi(f_1)\psi(f_2)$.

**Theorem 4.1.** The algebra homomorphisms are precisely given by

$$
(4.1) \quad \psi(f) = \int_G f(g)p(g) \, dg,
$$

where $C^\infty(K/G/K)$, $|p(g)| \leq p(1) = 1$, and

$$
(4.2) \quad p(g)p(h) = \int_K p(gh) \, dk.
$$

We call a function $p$ with these properties a spherical function. Note that since $K/G/K \cong [0, \pi]$, we can think of $f$ and $p$ as being functions of $\theta \in [0, \pi]$ or, equivalently, as depending on $\cos \theta$ only. If we take this point of view and integrate out the other variables, the above representation of $\psi$ becomes

$$
\psi(f) = \frac{1}{2} \int_0^\pi f(\cos \theta)p(\cos \theta) \sin \theta \, d\theta.
$$

**Sketch of proof.** The formal manipulation

$$
\psi(f_1)\psi(f_2) = \psi(f_1 * f_2) = \psi \left( \int_G f_1(gh^{-1})f_2(h) \, dh \right)
$$

$$
= \int_G \psi(f_1(gh^{-1}))f_2(h) \, dh
$$

makes it plausible that $\psi(f)$ has the integral representation given in the theorem (pick $f_1$ with $\psi(f_1) = 1$). It also seems reasonable to assume
that then \( p \) will be constant on double cosets and smooth. (These arguments can be made rigorous, of course.) We will now show that then (4.2) must hold for such a \( p \). We have that

\[
\int_G dg \int_G dh f_1(g) f_2(h) p(g)p(h) = \psi(f_1) \psi(f_2) = \psi(f_1 \ast f_2)
\]

\[
= \int_G dg p(g) \int_G dh f_1(gh^{-1}) f_2(h)
\]

\[
= \int_G dg \int_G dh f_1(g) f_2(h) p(gh).
\]

This does not imply that \( p(g)p(h) = p(gh) \) because \( f_1, f_2 \) are not arbitrary functions on \( G \): they are constant on double cosets. So, as in the remarks preceding the proof, we should first integrate out the other variables. This cannot be done directly because \( p(gh) \) need not be a function of the double cosets of \( g \) and \( h \) only. But the final integral is unchanged if we replace \( p(gh) \) by \( p(gkh) \) with \( k \in K \) (why?), and thus we can in fact replace \( p(gh) \) by the average \( \int_K p(gkh) \, dk \). This average is constant on \( K \, g \, K \) as well as on \( K \, h \, K \) (why?), so the argument outlined above now works and shows that (4.2) holds.

The condition that \( |p(g)| \leq 1 \) can be deduced from (4.2). We will also omit the proof of the converse, namely the assertion that every spherical function induces a homorphism by (4.1).

5. Spherical functions

We now want to analyze the spherical functions \( p \) in more detail. Most properties will follow from the fact that the spherical functions are eigenfunctions of the Laplacian

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\]

Exercise 5.1. Show that if a function \( f \) is expressed in spherical coordinates \( r, \theta, \varphi \), then

\[
\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_S f,
\]

where

\[
\Delta_S = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}
\]

is the spherical Laplacian. Warning: This is a rather tedious calculation, based on the chain rule.
The Laplace operator commutes with rotations. More precisely, for a (smooth, decaying) function $f$ on $\mathbb{R}^3$ and $g \in SO(3)$, let $(L_g f)(x) = f(gx)$.

**Exercise 5.2.** Prove that $L_g \Delta f = \Delta L_g f$.

**Hint:** Prove that both sides have the same Fourier transform. Recall that $(L_g f)^\wedge = L_g \hat{f}$.

Since rotations $g \in SO(3)$ act on the sphere $S^2$, it also makes sense to apply $L_g$ to functions $f : S^2 \to \mathbb{C}$. The definition still reads $(L_g f)(x) = f(gx)$ ($x \in S^2$).

**Exercise 5.3.** Deduce from the result of Exercises 5.1, 5.2 that $L_g \Delta_S f = \Delta_S L_g f$ for all $f \in C^\infty(S^2)$.

**Theorem 5.1.** Let $p$ be a spherical function, interpreted as a function on $S^2$ by using the identification $G/K \cong S^2$ from Theorem 1.2. Then $p$ is an eigenfunction of the spherical Laplacian: $\Delta_S p = \lambda p$.

**Proof.** In the identity (4.2), identify $h \in G$ with $x = hn \in S^2$ (it’s safe to do so because spherical functions are constant on double cosets). Apply $\Delta_S$ to both sides to obtain

$$p(g)\Delta_S p(x) = \int_K \Delta_S L_{gk} p(x) \, dk = \int_K L_{gk} \Delta_S p(x) \, dk,$$

or, going back to the original notation,

$$p(g)(\Delta_S p)(h) = \int_K (\Delta_S p)(gkh) \, dk.$$

Now $p$ and $\Delta_S p$ are constant on double cosets and $KgK = Kg^{-1}K$ (see Exercise 3.3), so

$$p(g)(\Delta_S p)(h) = p(g^{-1})(\Delta_S p)(h^{-1}) = \int_K (\Delta_S p)(g^{-1}kh^{-1}) \, dh$$

$$= \int_K (\Delta_S p)(hk^{-1}g) \, dk = \int_K (\Delta_S p)(hkg) \, dk$$

$$= p(h)(\Delta_S p)(g).$$

In particular, letting $h = 1$, we see that $\Delta_S p = \lambda p$, with $\lambda = (\Delta_S p)(1)$. \hfill $\square$

**Exercise 5.4.** Show that the spherical Laplacian is symmetric in the sense that $(\Delta_S f, g) = (f, \Delta_S g)$, where $f, g \in C^\infty(S^2)$ and $(f, g) = 1/(4\pi) \int_{S^2} f(x)\overline{g(x)} \, d\sigma(x)$.

**Hint:** Prove a similar result for $\Delta$ and deduce the claim from this.
Exercise 5.5. Show that eigenfunctions of $\Delta_S$ belonging to different eigenvalues $\lambda$ are orthogonal with respect to the scalar product introduced in the previous exercise.

*Hint:* Use the result of Exercise 5.4.

To actually determine those eigenfunctions of the spherical Laplacian that are constant on circles of latitude, we introduce the generating function

$$F(x, z) = (1 - 2xz + z^2)^{-1/2},$$

and expand into a power series in $z$:

$$F(x, z) = \sum_{n=0}^{\infty} p_n(x) z^n.$$

**Exercise 5.6.** Prove that for $|x| \leq 1$, $F$ is a holomorphic function of $z$ in $\{z \in \mathbb{C} : |z| < 1\}$.

We can get this power series by using the binomial series

$$(1 + y)^{-1/2} = \sum_{n=0}^{\infty} \left( \frac{-1/2}{n} \right) y^n.$$ 

In particular, this shows that $p_n$ is a polynomial of degree $n$, the $n$th Legendre polynomial. We will be interested in the functions $p_n(\cos \theta)$.

**Theorem 5.2.** $\Delta S p_n(\cos \theta) = -n(n+1)p_n(\cos \theta)$ and

$$\frac{1}{2} \int_0^{\pi} p_n^2(\cos \theta) \sin \theta d\theta = \frac{1}{2n+1}.$$

In other words, the $p_n$ are eigenfunctions of the spherical Laplacian, and they are constant on circles of latitude. By Exercise 5.5, they are also orthogonal. Indeed, with more work, one can show:

**Theorem 5.3.** The Legendre polynomials $p_n(\cos \theta)$ $(n \geq 0)$ are precisely the spherical functions. Moreover, $\{p_n(\cos \theta) : n \geq 0\}$ is an orthogonal basis of $L^2([0, \pi], \frac{1}{2} \sin \theta d\theta)$.

We will be satisfied with just proving Theorem 5.2.

**Proof of Theorem 5.2.** Note that for $0 \leq r < 1$, $F(\cos \theta, r)$ is the reciprocal of the distance $|x - n|$ between $x = r(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ and the north pole $n$.

**Exercise 5.7.** Show that $\Delta |x - x_0|^{-1} = 0$ for $x \neq x_0$. 
By Exercises 5.7 and 5.1
\[ 0 = \Delta F(\cos \theta, r) = \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S \right) F(\cos \theta, r) \]
\[ = \sum_{n=0}^{\infty} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S \right) p_n(\cos \theta) r^n \]
\[ = \sum_{n=0}^{\infty} (n(n+1)p_n(\cos \theta) + \Delta_S p_n(\cos \theta)) r^{n-2}. \]
This implies the first formula from Theorem 5.2.

The functions \( p_n(\cos \theta) \), being eigenfunctions of \( \Delta_S \), are thus orthogonal by Exercise 5.5. In particular, for real \( z \in (-1, 1) \),
\[ \frac{1}{2} \int_{0}^{\pi} |F(\cos \theta, z)|^2 \sin \theta \, d\theta = \sum_{n=0}^{\infty} \|p_n\|^2 z^{2n}. \]
The integral on the left-hand side can be evaluated explicitly:
\[ \frac{1}{2} \int_{0}^{\pi} \frac{\sin \theta}{1 - 2z \cos \theta + z^2} \, d\theta = \frac{1}{2} \int_{-1}^{1} \frac{dx}{1 - 2zx + x^2} \]
\[ = \frac{-1}{4z} \ln(1 - 2zx + z^2) \bigg|_{-1}^{1} \]
\[ = \frac{-1}{4z} \ln \frac{1 - 2z + z^2}{1 + 2z + z^2} \]
\[ = \frac{1}{2z} \ln \frac{1 + z}{1 - z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n + 1} \]
In the last step, we use the power series expansion
\[ \ln(1 + y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n. \]

Let us summarize what we have accomplished so far: There are homomorphisms mapping functions on \( G/K/G \) to \( \mathbb{C} \); they correspond to the spherical function \( p_n(\cos \theta) \), or, equivalently, to the eigenfunctions of the spherical Laplacian that depend on latitude only. In fact, such homomorphisms exist in sufficiently large supply and we can expand every (integrable, say) function on \( G/K/G \) into a generalized Fourier series:
\[ f = \sum_{n=0}^{\infty} \hat{f}(n)p_n \]
where
\[ \frac{\hat{f}(n)}{2n + 1} = \int_{S^2} f p_n \, d\sigma = \frac{1}{2} \int_{0}^{\pi} f(\cos \theta)p_n(\cos \theta) \sin \theta \, d\theta = \int_{G} f p_n \, dg. \]
6. Spherical harmonics

We will now extend the theory to functions on $G/K \cong S^2$. We then need additional functions on the sphere, not necessarily constant on circles of latitude. We obtain these new functions by letting $G$ act on the $p_n$. More precisely, define $p^g_n(x) = p_n(gx)$ ($g \in G$, $x \in S^2$) and let $M_n$ be the space spanned by \{ $p^g_n : g \in G$ \}. (More precisely, $M_n$ is the closed subspace of $L^2(S^2)$ spanned by the $p^g_n$. We will not be very precise about this in the sequel and also leave convergence issues aside. As it happens, the $M_n$ turn out to be finite dimensional so that actually there are no such problems anyway.)

The spherical harmonics (of weight $n$) are, by definition, the functions from $M_n$. Let $Y_{nl}$, $l = 0, \pm 1, \ldots$ be an ONB of $M_n$.

Note that we are, as usual, not very concerned about properly distinguishing between group elements, points on the sphere, and latitude. For instance, to actually evaluate $p_n(gx)$, we would have to apply $g \in G$ to $x \in S^2$ and determine the latitude $\theta$ of the resulting point $gx \in S^2$ to obtain $p_n(gx)$ as $p_n(\cos \theta)$, this being one of the functions from the previous section. To make things worse, we might also write $p_n(gh)$ instead; in this case, we first identify $h \in G$ with the point $x = hn \in S^2$ and then proceed as above.

Since $\Delta_S$ commutes with the action $L_g$ of $G$ on functions on the sphere (compare Exercise 5.3), the $Y_{nl}$ are still eigenfunctions of $\Delta_S$ with eigenvalue $-n(n+1)$. Now expand $p^g_n$, using the basis $\{ Y_{nl} \}$:

$$p^g_n(x) = \sum_l c^g_n(l) Y_{nl}(x),$$

with unknown coefficients $c^g_n(l) \in \mathbb{C}$. We can determine the $c^g_n(l)$ by looking at the scalar product $(p^g_n, p^g_n')$. Using the fact that the $p_n$’s are constant on double cosets and invariance of the Haar measure, we obtain that

\[
\int_G p^g_n(h)p^g_n(h) \, dh = \int_G p_n(gh)p_n(g'h) \, dh = \int_G p_n(k^{-1}gh)p_n(g'h) \, dh \\
= \int_G p_n(h)p_n(g'g^{-1}kh) \, dh.
\]
This can now be integrated over $K$; we also use formula (4.2):

$$
\int_G p_n^g(h)p_n^g(h) \, dh = \int_G dh \, p_n(h) \int_K dk \, p_n(g'g^{-1}kh)
= \int_G dh \, p_n(h)p_n(g'g^{-1})p_n(h)
= p_n(g'g^{-1})\|p_n\|^2 = \frac{p_n'(g^{-1})}{2n+1}
$$

By taking linear combinations of this formula, we in fact see that

$$
\int_G p_n^g(h)f(h) \, dh = \frac{f(g^{-1})}{2n+1}
$$

for all $f \in M_n$. In particular, choosing $f = Y_{nl}$, we obtain that $c_n^g(l) = Y_{nl}(g^{-1})/(2n+1)$. We plug this back into (6.1), replace $x$ by $h$ and $g$ by $g^{-1}$, and summarize:

**Theorem 6.1.** The spherical harmonics satisfy the addition formula:

$$
\frac{1}{2n+1} \sum_l Y_{nl}(g)Y_{nl}(h) = p_n(g^{-1}h)
$$

As a consequence, we obtain:

**Corollary 6.1.** $\dim M_n = 2n + 1$

**Proof.** With $g = h$, the addition formula says that $(2n+1)^{-1} \sum |Y_{nl}(g)|^2 = p_n(1) = 1$, and since $\|Y_{nl}\| = 1$, integration over $G$ now shows that there must be exactly $2n + 1$ summands. $\square$

We label so that $l$ varies over $-n, \ldots, n$. Again, there is a completeness result (compare Theorem 5.3): The $Y_{nl}$ ($n \geq 0$, $-n \leq l \leq n$) form an ONB of $L_2(S^2)$. So every function $f \in L_2(S^2)$ can be expanded as

$$
f(x) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} c_{nl}Y_{nl}(x),
$$

with $c_{nl} = (f, Y_{nl})$. Moreover, $M_n$ is precisely the space of eigenfunctions of $\Delta_S$ with eigenvalue $-n(n+1)$.

### 7. Representations of $SO(3)$

As the final step, it remains to extend the theory from functions on $S^2 \cong G/K$ to functions on $G$. Motivated by the treatment of
the preceding section, we let $G$ act on the spherical harmonics and introduce coefficients $U_{ij}^n(g)$ by writing

$$Y_{ni}(gx) = \sum_{j=-n}^{n} U_{ij}^n(g) Y_{nj}(x).$$

Such a representation of $Y_{ni}(gx)$ is possible because this function is in the eigenspace of $\Delta_S$ belonging to the eigenvector $-n(n+1)$ (Exercise 5.5 again!) and the $Y_{nj} (-n \leq j \leq n)$ span this space. Write $U_n(g)$ for the $(2n+1) \times (2n+1)$ matrix with entries $U_{ij}^n(g)$.

**Theorem 7.1.** $U_n(g)$ is unitary ($U_n^* U_n = 1$) and $U_n(g)U_n(h) = U_n(gh)$ for all $g, h \in G$.

In other words, the map $g \mapsto U_n(g)$ is a homomorphism from $G$ to $U(2n+1)$, the group of unitary matrices on $\mathbb{C}^{2n+1}$. Such a homomorphism from a group to a matrix group is called a representation of $G$. So, using this term, we have discovered representations of $SO(3)$. More importantly, these representations are the building blocks for the harmonic analysis of functions on $G$; they take the role of the characters in the abelian case.

**Proof of Theorem 7.1.** To check that $U_n(g)$ is unitary, use (7.1) to evaluate

$$\delta_{ij} = \frac{1}{4\pi} \int_{S^2} Y_{ni}(x)Y_{nj}(x) \, d\sigma(x) = \frac{1}{4\pi} \int_{S^2} Y_{ni}(gx)Y_{nj}(gx) \, d\sigma(x).$$

This yields

$$\delta_{ij} = \sum_{k,l=-n}^{n} U_{ik}^n(g)U_{lj}^n(g) \frac{1}{4\pi} \int_{S^2} Y_{nk}(x)Y_{nl}(x) \, d\sigma(x)$$

$$= \sum_{k=-n}^{n} U_{ik}^n(g)U_{jk}^n(g) = (U_nU_n^*)_{ji},$$

as claimed (recall that for matrices $A$, $B$, we have that $AB = 1$ if and only if $BA = 1$).

To verify the homomorphism property, compute $Y_{ni}(ghx)$ in two ways:

$$Y_{ni}(ghx) = \sum_{j=-n}^{n} U_{ij}^n(gh) Y_{nj}(x) = \sum_{k=-n}^{n} U_{ik}^n(g) U_{nk}^n(hx)$$

$$= \sum_{k=-n}^{n} U_{ik}^n(g) \sum_{j=-n}^{n} U_{kj}^n(h) Y_{nj}(x).$$
Since the $Y_{nj}$ ($|j| \leq n$) are linearly independent, it follows that

$$U_n^{ij}(gh) = \sum_{k=-n}^{n} U_n^{ik}(g) U_n^{kj}(h) = (U_n(g) U_n(h))_{ij},$$

as required.

We will conclude this section by describing (without proofs) the use of these representations for the harmonic analysis of functions on $G$. For $f \in L_2(G)$, define

$$\hat{f}(n) = \int_{G} f(g) U_n^*(g) \, dg.$$ 

Note that $\hat{f}(n)$ is a $(2n + 1) \times (2n + 1)$ matrix. We then have that

$$f(g) = \sum_{n=0}^{\infty} (2n + 1) \text{tr} \left( \hat{f}(n) U_n(g) \right)$$

(“Fourier inversion”) and

$$\int_{G} |f(g)|^2 \, dg = \sum_{n=0}^{\infty} (2n + 1) \text{tr} \left( \hat{f}(n) \hat{f}(n)^* \right)$$

(“Plancherel identity”). Here, $\text{tr} M$ denotes the trace of the matrix $M$, that is, $\text{tr} M = \sum M_{ii}$.

**Exercise 7.1.** Prove that $(f_1 * f_2)(n) = \hat{f}_2(n) \hat{f}_1(n)$ (in this order!).