DYNAMICAL SYSTEMS

CHRISTIAN REMLING

We will discuss two topics from the theory of dynamical systems: dynamics of the logistic map and Sarkovski's Theorem.

1. Basic notions

A discrete time dynamical system consists a *phase space* X and a map $f : X \to X$. By repeatedly applying f, we obtain the *orbit* of a point $x \in X$:

$$(x, f(x), f(f(x)), \ldots)$$

We think of this sequence as describing the time evolution of the point x. It will be convenient to denote the n-fold composition of f with itself by f^n, so $f^n(x)$ is the result of n applications of f to x.

In the theory of dynamical systems, we are typically interested in the long time behavior of orbits. This can depend very sensitively on small details, even in simple examples.

As an illustration, consider the rotation by α on the unit circle $S^1 = \{e^{i\varphi} : 0 \le \varphi < 2\pi\}$. So $f(e^{i\varphi}) = e^{i(\varphi+\alpha)}$.

Exercise 1.1. Show that this system has periodic orbits precisely if $\alpha/(2\pi) \in \mathbb{Q}$.

We call a point $x \in X$ *periodic* if $f^n(x) = x$ for some $n \ge 1$. The smallest such n is called the *period* of x.

On the other hand, if $\alpha/(2\pi) \notin \mathbb{Q}$, then, roughly speaking, the orbit of an arbitrary $e^{i\varphi}$ fills out the whole unit circle; it is dense.

Exercise 1.2. Show this. More precisely, show that given any $x, y \in S^1$ and $\epsilon > 0$, there exists $n \ge 0$ so that $|f^n(x) - y| < \epsilon$.

It can in fact be shown that subsets I of the unit circle get their fair share $\ell(I)/(2\pi)$ of orbit points if the system is run long enough; here, $\ell(I)$ denotes the arc length of I.

2. One-dimensional systems

By this we mean system with phase space $X = \mathbb{R}$, or a subset of $\mathbb{R}$. We also assume for convenience that $f \in C^\infty(\mathbb{R})$.

A *fixed point* is a point p with $f(p) = p$ (equivalently, a periodic point with period 1). Let's try to find the fixed points of the simple example $f(x) = x^3$. These are the points that satisfy $x^3 = x$, so the fixed points are $x = 0$ and $x = \pm 1$. These points do not get moved at all by the dynamics. If $0 < x < 1$, then we claim that $f^n(x) \to 0$. To prove this carefully, the following fact is useful.

Theorem 2.1. *If $f^n(x) \to x_0$, then $f(x_0) = x_0$.*

Proof. Since f is continuous, we have that

$$f(x_0) = \lim_{n\to\infty} f(f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x_0.$$

□

So orbits can only converge to fixed points.

Let's now look again at $f^n(x)$ for $0 < x < 1$. Clearly, the sequence is decreasing and bounded below by 0. Thus it converges to some limit, which must lie in $[0, 1)$. Theorem 2.1 says that the limit has to be a fixed point, and the only fixed point in this range is 0. This proves our claim.

Similarly, one can show that $f^n(x) \to \infty$ if $x > 1$. Also, $f^n(x) \to 0$ for $-1 < x < 0$ and $f^n(x) \to -\infty$ for $x < -1$.

If we review the whole situation, it appears that 0 attracts orbits while the other two fixed points ± 1 repel them. We can give a precise definition that captures this intuition. We call a fixed point p *stable* if for every $\epsilon > 0$, there exists $\delta > 0$ so that $|f^n(x) - p| < \epsilon$ for all $n \geq 0$, provided that $|x - p| < \delta$. We also introduce the *stable set* of p

$$W^s(p) = \{x : f^n(x) \to p\}.$$

We then call p an *attracting fixed point* if p is stable and $W^s(p) \supset (p - \epsilon, p + \epsilon)$ for some $\epsilon > 0$. There is a similar notion of repelling fixed points as those points that attract orbits under backward time evolution, but we don't want to make this precise (note that if f is not invertible, it's not clear how one can unambiguously go back in time).

In the above example, $W^s(0) = (-1, 1)$ and 0 is stable, so it is an attracting fixed point. The other two fixed points are repelling.

Theorem 2.2. *Suppose that p is a fixed point with $|f'(p)| < 1$. Then p is attracting.*

Proof. Since f' is continuous, we can find an interval $[p - d, p + d]$ so that $|f'| \leq q < 1$ on this interval. Then, if $x \in [p - d, p + d]$, the mean value theorem shows that

$$|f(x) - p| = |f'(t)|\, |x - p| \leq q|x - p|.$$

In particular, $f(x)$ lies in the same interval and we can repeat this argument. By iterating, we find that $|f^n(x) - p| \leq q^n |x - p|$. This shows that p is stable and $W^s(p) \supset (p - d, p + d)$. $\square$

Similarly, it can be shown that p will be repelling if $|f'(p)| > 1$.

Exercise 2.1. (a) Find the fixed points of the map

$$f(x) = \frac{1}{2}(x^2 - x).$$

Which of these are attracting? Find W^s for the attracting fixed point(s).
(b) Compute $f^2(x)$. Then show that f has no periodic points of period 2.

Exercise 2.2. Consider the *tent map*

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & 1/2 \leq x \leq 1 \end{cases}$$

on $X = [0, 1]$.
(a) Sketch the graph of f and f^2. What does the graph of f^n look like? Use this information to conclude that f^n has exactly 2^n fixed points (equivalently, the system has exactly 2^n periodic points whose period is a divisor of n). Show that the periodic points are dense in $[0, 1]$.
(b) Show that $f^n(x) = 2N_n \pm 2^n x$, where N_n is an integer. Conclude that $x \notin \mathbb{Q}$ is never periodic, so the non-periodic points are also dense.

3. The logistic map

This is the name given to the map

$$f(x) = f_\mu(x) = \mu x(1 - x),$$

with $\mu > 0$. It's easy to check that f has two fixed points, 0 and $p_\mu = (\mu - 1)/\mu$. Moreover, $f'(0) = \mu$, $f'(p_\mu) = 2 - \mu$, so for $\mu > 3$, both fixed points are repelling. The fact that $|f'| > 1$ at the fixed points also implies that $f^n(x)$ cannot converge if $x \neq 0, p_\mu$.

Exercise 3.1. Prove this carefully.

Proposition 3.1. *Suppose that $\mu > 1$. If $x \notin [0, 1]$, then $f^n(x) \to -\infty$.*

Proof. If $x < 0$, then

$$f(x) = \mu x - \mu x^2 < \mu x < x,$$

so the sequence $f^n(x)$ is decreasing. Thus, it will either approach a limit or diverge to $-\infty$. The first case is impossible because there are no negative fixed points.

If $x > 1$, then $f(x) < 0$. $\square$

4. Conjugacy of systems

The notion of an isomorphism is fundamental in any mathematical theory. One thinks of isomorphic objects as different concrete realizations of the same structure.

For example, the structure of a topological space is, by definition, given by the collection of its open sets, and thus the appropriate definition of an isomorphism asks for a bijective map that maps precisely the open sets to open sets again (it does not do anything really but change names). In this context, the isomorphism are called *homeomorphisms.*

Now suppose we are given two dynamical systems (X, f), (Y, g), with phase spaces X, Y that are topological spaces (we will then also insist that f, g are continuous maps).

Definition 4.1. We call (X, f), (Y, g) *(topologically) conjugate* if there exists a homeomorphism $\varphi : X \to Y$ so that $f = \varphi^{-1} \circ g \circ \varphi$. We then call φ a *(topological) conjugacy* between these systems.

Again, we can interpret conjugate systems as different realizations of the same (topological) dynamical system. In particular, all properties of one system carry over to the other system. For example:
(a) $x \in X$ is a periodic point of period n if and only if $\varphi(x)$ is periodic of period n.
(b) $x \in X$ is an attracting fixed point if and only if $\varphi(x)$ is an attracting fixed point.

Exercise 4.1. Prove these statements.

Exercise 4.2. Show that the rotations $T_\alpha(z) = ze^{i\alpha}$ and $T_{-\alpha}$ define conjugate systems on the unit circle. Can you also provide an example where T_α, T_β are not conjugate?

5. The invariant Cantor set for the logistic map

We now return to the map $f(x) = \mu x(1 - x)$, for $\mu > 4$. We will in fact focus on $I = [0, 1]$. We saw above that $f^n(x) \to -\infty$ as soon as $f^{n_0}(x) \notin I$ for some n_0. Are there any points whose orbits will stay in I forever? This is answered by the following result.

Theorem 5.1. *Assume that* $\mu > 4$*, and define*

$$Y = \{x \in I : f^n(x) \in I \text{ for all } n \geq 0\}.$$

Then Y *is a (= homeomorphic to the) Cantor set.*

We will in fact only prove this under the somewhat stronger assumption that $\mu > 2 + \sqrt{5}$; this will simplify one part of the proof.

Proof. By direct inspection of f, we see that $f^{-1}(I) = I_1 \cup I_2$, with two disjoint intervals $I_1 = [0, x_-]$, $I_2 = [x_+, 1]$. (A calculation shows that $x_\pm = 1/2 \pm \sqrt{1/4 + 1/\mu}$, but we won't need these formulae.)

Clearly, $x, f(x) \in I$ if and only if $x \in I_1 \cap I_2$. Similarly, we will have that, in addition, $f^2(x) \in I$ precisely if $x \in \bigcup I_{j_0 j_1}$, where

$$I_{j_0 j_1} = \{x \in I : x \in I_{j_0}, f(x) \in I_{j_1}\}.$$

More generally, we define

$$I_{j_0 \cdots j_n} = \{x \in I : x \in I_{j_0}, \ldots, f^n(x) \in I_{j_n}\}$$

and

$$C_n = \bigcup_{j_0,\ldots,j_n=1,2} I_{j_0 \cdots j_n}.$$

It is clear that $Y = \bigcap C_n$, and we're hoping that this displays Y as a Cantor set type intersection of collections of disjoint intervals. To confirm this, we need some preparations.

Lemma 5.2. *(a)* $C_n \cap I_{j_0 \cdots j_{n-1}} = I_{j_0 \cdots j_{n-1} 1} \cup I_{j_0 \cdots j_{n-1} 2}$ *is a disjoint union of two non-empty closed intervals.*
(b) If $(j_0, \ldots, j_n) \neq (j_0', \ldots, j_n')$*, then the intervals* $I_{j_0 \cdots j_n}$*,* $I_{j_0' \cdots j_n'}$ *are disjoint. Thus* C_n *is a disjoint union of* 2^{n+1} *intervals.*
(c) f *maps* $I_{j_0 \cdots j_n}$ *homeomorphically onto* $I_{j_1 \cdots j_n}$*.*

Exercise 5.1. Find the relative location of the intervals $I_{j_0 j_1 j_2}$, $j_k = 1, 2$.

Proof of Lemma 5.2. Notice that C_n is the set of all $x \in I$ for which $x, f(x), \ldots, f^n(x) \in I_1 \cup I_2$. Moreover, a fixed subset $I_{j_0 \cdots j_n} \subset C_n$ contains exactly those $x \in C_n$ for which the orbit follows the corresponding itinerary

$$x \in I_{j_0}, \ldots, f^n(x) \in I_{j_n}.$$

From the definition, we also have that

$$I_{j_0 \cdots j_n} = I_{j_0} \cap f^{-1}(I_{j_1 \cdots j_n}). \tag{5.1}$$

Now if $J \subset [0, 1]$ is a closed interval, then $f^{-1}(J)$ is a disjoint union of two closed intervals, one contained in I_1 and one in I_2. This we see by looking at the graph of f. As a consequence, (5.1) now indeed shows that all $I_{j_0 \cdots j_n}$ are closed intervals (use induction on n!).

Statements (a) and (b) follow now from these observations. As f, restricted to I_j for $j = 1$ or $j = 2$, is injective, (5.1) also shows that (c) holds. $\square$

In particular, the sets C_n are closed, and hence so is $Y = \bigcap C_n$. Since $Y \subset [0, 1]$, this in fact shows that Y is compact. We want to show that Y is also perfect (no isolated points) and totally disconnected (does not contain an interval). This will be a consequence of the fact that the lengths of the subintervals of C_n go to zero as $n \to \infty$:

$$\lim_{n\to\infty} \max_{j_0,\ldots,j_n} |I_{j_0\cdots j_n}| = 0 \tag{5.2}$$

Exercise 5.2. Show that (5.2) indeed implies that $\bigcap C_n$ is perfect and totally disconnected.

To establish (5.2), we will make use of the following estimate:

Lemma 5.3. *Suppose that $\mu > 2 + \sqrt{5}$. Then $\min_{x\in I_1\cup I_2} |f'(x)| > 1$.*

Proof of Lemma 5.3. We have that $f'(x) = \mu(1-2x)$ and $f''(x) = -2\mu < 0$, so the smallest value of $|f'|$ on $I_1 \cup I_2$ occurs at x_+ or x_-. At these points, $f(x_\pm) = 1$, so a calculation shows that $x_\pm = 1/2 \pm \sqrt{1/4 - 1/\mu}$. Hence

$$|f'(x_\pm)| = \sqrt{\mu^2 - 4\mu}.$$

This last expression is strictly large than 1 if $\mu > 2+\sqrt{5}$, as claimed. $\square$

Let $q = \min_{I_1\cup I_2} |f'|$. Then, by the Lemma, $q > 1$. Moreover, Lemma 5.2(c), combined with the mean value theorem shows that

$$|I_{j_1\cdots j_n}| = |f'(c)|\, |I_{j_0\cdots j_n}| \ge q\, |I_{j_0\cdots j_n}|\,.$$

By iterating this inequality, we obtain that

$$|I_{j_0\cdots j_n}| \le Cq^{-n},$$

and this gives (5.2). $\square$

6. Symbolic dynamics on the invariant Cantor set

We call the Cantor set Y *invariant* because it has the property that $f(y) \in Y$ if $y \in Y$. In particular, we can restrict our attention to Y; we obtain a new (smaller) dynamical system (Y, f).

We denote by X_2 the space $\{1,2\}^{\mathbb{N}}$ of sequences $x = (x_1, x_2, \ldots)$ on two symbols $x_j = 1, 2$. We also define a metric on X_2, as follows:

$$d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n| 3^{-n}$$

We already know that (X_2, d) is a compact space; in fact, it is homeomorphic to the Cantor set. The shift $(Sx)_n = x_{n+1}$ is a continuous map on X_2.

Exercise 6.1. Show this.

From the material of the previous section, we have a natural map from Y to X_2: we map a point y to its itinerary $j_0, j_1, \ldots$. We call this the *itinerary map* and denote it by $h : Y \to X_2$.

Theorem 6.1. *The itinerary map provides a topological conjugacy between the systems* (Y, f) *and* (X_2, S).

Recall that this means that h has the following properties:
(1) $h : Y \to X_2$ is a homeomorphism.
(2) $f = h^{-1} \circ S \circ h$

Proof. Property (2) is clear from the construction once we have (1). Let's first check that h is onto. So let $x = (x_1, x_2, \ldots) \in X_2$. By Lemma 5.2(a), the intervals $I^{(n)} = I_{x_1 x_2 \ldots x_n}$ are all closed and non-empty, and they are nested: $I^{(1)} \supset I^{(2)} \supset \ldots$ Thus their intersection $\bigcap I^{(n)}$ is non-empty, and any $t \in \bigcap I^{(n)}$ satisfies $\varphi(t) = x$ (t is in fact unique).

Next, we show that h is injective. We again do this only under the slightly stronger assumption that $\mu > 2 + \sqrt{5}$. We saw above, in Lemma 5.3, that then $q = \min_{I_1 \cup I_2} |f'| > 1$. This implies that $|f(x) - f(y)| \geq q|x - y|$ if x, y both lie in the same interval I_j for $j = 1$ or 2. Now $h(x) = h(y)$ says that $f^n(x)$, $f^n(y)$ in fact are in the same interval I_{x_n} for all n, thus $|f^n(x) - f^n(y)| \geq q^n |x - y|$. Since $|f^n(x) - f^n(y)| \leq 1$, this implies that $x = y$.

Finally, to show that h is continuous, we only need to observe that if $f^n(x) \in I_1$, say, then $f^n(y) \notin I_2$ for all y sufficiently close to x. Since either $f^n(x) \in I_1$ or $f^n(x) \in I_2$ for all n if $x \in Y$, this shows that if $x \in Y$ and $N \in \mathbb{N}$ are given, we can find an $\epsilon > 0$ so that the first N entries of $h(y)$ agree with those of $h(x)$ for all $y \in Y$, $|y - x| < \epsilon$. It follows that

$$d(h(y), h(x)) \leq \sum_{n > N} 3^{-n} = 2 \cdot 3^{-N}.$$

This verifies that h is continuous, and as h is a bijection between compact metric spaces, continuity of the inverse is now automatic. □

This representation of (Y, f) as a shift on two symbols gives a lot of interesting information about the system:

Theorem 6.2. *(a)* f^n *has exactly* 2^n *fixed points.*
(b) The periodic points are dense in Y.
(c) There exist points $y \in Y$ *whose orbit is dense in* Y*; in fact, the set of such points is itself dense in* Y.
(d) There exists a constant $d > 0$ *such that the following holds: for any* $y \in Y$ *and* $\epsilon > 0$, *there exists* $y' \in Y$ *with* $|y - y'| < \epsilon$ *so that* $|f^n(y) - f^n(y')| \geq d$ *for all* $n \geq N = N(y, \epsilon)$.

Proof. We will work with the symbolic representation (X_2, S) of (Y, f).

(a) Notice that $x \in X_2$ is a fixed point of S^n precisely if the initial block $(x_1 \ldots x_n)$ is repeated indefinitely. There are 2^n such blocks of length n.

(b) Again, we can establish this by showing that (X_2, S) has the same property. If $x \in X_2$ is given, let $y_n = x_n$ for $1 \le n \le N$ and then continue periodically to define y_n for $n > N$. Then y is periodic with period $\le N$, and

$$d(x, y) \le \sum_{n>N} 3^{-n} = (1/2)3^{-N}.$$

N was arbitrary here, so there are periodic points arbitrarily close to x, as claimed.

(c) Let

$$y = (0|1|00|01|10|11|000|\ldots);$$

this point has a dense orbit because for any given $x \in X_2$ and $N \in \mathbb{N}$, a suitable shift $S^n y$ will agree with x on an initial piece of length $\ge N$.

More generally, we can start with an arbitrary finite block and then construct the remaining digits as above, to obtain a point with a dense orbit that is close to a given point.

(d) We can take $d > 0$ as the length $x_+ - x_-$ of the first gap. Then $|x - y| \ge d$ whenever the first digits of the symbolic representations $h(x)$, $h(y)$ differ from each other. Given $y \in Y$, we can now obtain y' as follows: Consider the symbolic representation $x = h(y)$ of y, let $x'_n = x_n$ for $n = 1, \ldots, N$ and $x'_n \ne x_n$ for $n > N$. Then, as above, $d(x, x') = (1/2)3^{-N}$, so $|y - y'|$ will be small as well by the continuity of h^{-1}. Moreover, by construction, $|f^n(y) - f^n(y')| \ge d$ for all $n > N$. $\square$

Exercise 6.2. Prove in similar style that (Y, f) is *topologically mixing:* If $x, y \in Y$ and $N \in \mathbb{N}$, $\epsilon > 0$ are given, then there are $x' \in Y$ and $n \ge N$ so that $|x - x'| < \epsilon$, $|f^n(x') - y| < \epsilon$.

Exercise 6.3. How many periodic points of exact (!) period 6 does the system $(\mathbb{R}, f)$ have?

Exercise 6.4. Are there points $y \in Y$ that are not periodic and whose orbit is not dense in Y?

7. Sarkovskii's Theorem

To state Sarkovskii's Theorem in full generality, we need to introduce the Sarkovskii ordering of the positive integers:

$$3 \succ 5 \succ 7 \succ \ldots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ \ldots \succ 2^2 \cdot 3 \succ \ldots \succ$$
$$2^n \cdot 3 \succ 2^n \cdot 5 \succ \ldots \succ 2^{n+1} \succ 2^n \succ \ldots \succ 2 \succ 1$$

In 1964, Sarkovskii proved the following striking result:

Theorem 7.1 (Sarkovskii)**.** *Suppose that the continuous map $f : \mathbb{R} \to \mathbb{R}$ has a point of period n and $n \succ k$. Then f has a point of period k.*

We will discuss only a special case ("period 3 implies chaos") of Sarkovskii's Theorem, but this, too, is quite spectacular:

Corollary 7.2. *Suppose that the continuous map $f : \mathbb{R} \to \mathbb{R}$ has a point of period 3. Then f has periodic points of all periods.*

Proof. Let a be a point of period 3. We will assume that specifically

$$a = f^3(a) < f(a) < f^2(a).$$

Please convince yourself that up to reflection and replacing a with $f^k(a)$, this is in fact the general case.

Let $I_1 = [a, f(a)]$, $I_2 = [f(a), f^2(a)]$. By the intermediate value theorem, $f(I_1) \supset I_2$. It will be convenient to introduce a special notation for this kind of situation. We say that an interval I *f-covers* another interval J if $f(I) \supset J$; in this case, we write $I \to J$.

Lemma 7.3. *Let I, J be closed intervals and suppose that $I \to J$. Then there exists a closed subinterval $K \subset I$ so that $f(K) = J$, $f(\partial K) = \partial J$, $f(int(K)) = int(K)$.*

Proof of Lemma 7.3. Write $J = [b_1, b_2]$. Fix $a_1, a_2 \in I$ with $f(a_j) = b_j$. Let's assume that $a_1 < a_2$, the other case being similar. Let

$$x_1 = \sup\{x \in [a_1, a_2] : f(x) = b_1\}.$$

Then $f(x_1) = b_1$, by continuity, and $f(x) > b_1$ for $x > b_1$, from the definition of x_1. In particular, $x_1 < a_2$, so we can now define

$$x_2 = \inf\{x \in [x_1, a_2] : f(x) = b_2\}.$$

As before, we notice that $f(x_2) = b_2$ and $f(x) < b_2$ for $x < x_2$. Thus the interval $K = [x_1, x_2]$ has the desired properties. □

Lemma 7.4. *Suppose that the compact interval I f-covers itself. Then f has a fixed point in I.*

Proof of Lemma 7.4. Use the previous lemma to find a subinterval $K = [a, b] \subset I = [c, d]$ with $f(K) = I$ and either:
(i) $f(a) = c \leq a$, $f(b) = d \geq b$, or
(ii) $f(a) = d > a$, $f(b) = c < b$.

In both cases, the intermediate value theorem shows that $g(x) = f(x) - x$ has a zero in $[a, b]$. □

Lemma 7.5. *Suppose that $J_0 \to J_1 \to \ldots \to J_n = J_0$. Then f^n has a fixed point x_0, and $f^k(x_0) \in J_k$ for $k = 0, 1, \ldots, n$.*

Note that the intervals J_k need not be distinct here.

The existence of a fixed point x_0 of f^n is an immediate consequence of Lemma 7.4 since we certainly have that $f^n(J_0) \supset J_0$. To obtain the somewhat more precise statement of the Lemma, we need to take a closer look at the situation.

Proof of Lemma 7.5. We make the following claim, which we will prove by induction on j: *There exists a subinterval $K_j \subset J_0$ such that $f^k(K_j) \subset J_k$, $f^k(int(K_j)) \subset int(J_k)$ for $k = 1, \dots, j$ and $f^j(K_j) = J_j$.*

For $j = 1$, this is Lemma 7.3. Now assume the statement holds for $j-1$, that is, there exists a subinterval $K_{j-1} \subset J_0$ with the above properties. In particular, $f^j(K_{j-1}) = f(J_{j-1}) \supset J_j$, so Lemma 7.3, applied to f^j, lets us find a subinterval $K_j \subset K_{j-1}$ such that $f^j(K_j) = J_j$ and $f^j(\text{int}(K_j)) = \text{int}(J_j)$. This interval K_j has the desired properties.

In particular, $f^n(K_n) = J_0$, and, as observed above, we now obtain the existence of a fixed point $x_0 \in K_n$ of f^n from Lemma 7.5. By construction, $f^j(x_0) \in J_j$. □

We can now finish the proof of Corollary 7.2 as follows. Notice that $I_1 \to I_2$ and $I_2 \to I_1$, $I_2 \to I_2$. In particular, this last fact implies the existence of a fixed point. If $n \geq 2$, $n \neq 3$ is given, we consider the loop

$$I_1 \to I_2 \to \; I_2 \to I_2 \to \dots \to I_2 \to I_1,$$

with $n-1$ copies of I_2 in the middle. By Lemma 7.5, there exists $x_0 \in I_1$ with $f^n(x_0) = x_0$ and $f^j(x_0) \in I_2$ for $j = 1, \dots, n-1$. We claim that x_0 has exact period n. Indeed, if $f^k(x_0) = x_0$ for some $k < n$, then it would follow that $x_0 \in I_1 \cap I_2$, so $x_0 = f(a)$. However, we know that this point has period 3, so this is only possible if $n = 3k$, but even in this case we obtain a contradiction to the pattern ($f^j(x_0) \in I_2$) that was observed above. □