8. Commutative Banach Algebras

In this chapter, we analyze commutative Banach algebras in greater detail. So we always assume that $xy = yx$ for all $x, y \in A$ here.

**Definition 8.1.** Let $A$ be a (commutative) Banach algebra. A subset $I \subset A$ is called an *ideal* if $I$ is a (linear) subspace and $xy \in I$ whenever $x \in I$, $y \in A$. An ideal $I \neq A$ is called *maximal* if the only ideals $J \supset I$ are $J = I$ and $J = A$.

Ideals are important for several reasons. First of all, we can take quotients with respect to ideals, and we again obtain a Banach algebra.

**Theorem 8.2.** Let $I \neq A$ be a closed ideal. Then $A/I$ is a Banach algebra.

This needs some clarification. The quotient $A/I$ consists of the equivalence classes $(x) = x + I = \{x + y : y \in I\}$, and we define the algebraic operations on $A/I$ by working with representatives; the fact that $I$ is an ideal makes sure that everything is well defined (independent of the choice of representative). Since $I$ is in particular a closed subspace, we also have the quotient norm available, and we know from Theorem 2.18 that $A/I$ is a Banach space with this norm. Recall that this norm was defined as

$$
\|(x)\| = \inf_{y \in I} \|x + y\|.
$$

**Proof.** From the above remarks, we already know that $A/I$ is a Banach space and a commutative algebra with unit $(e)$. We need to discuss conditions (3), (4) from Definition 7.1. To prove (4), let $x_1, x_2 \in A$, and let $\epsilon > 0$. We can then find $y_1, y_2 \in I$ so that $\|x_j + y_j\| < \|(x_j)\| + \epsilon$. It follows that

$$
\|((x_1)(x_2))\| = \|(x_1x_2)\| \leq \|(x_1 + y_1)(x_2 + y_2)\|
\leq \|x_1 + y_1\| \|x_2 + y_2\| \leq (\|(x_1)\| + \epsilon)(\|(x_2)\| + \epsilon).
$$

Since $\epsilon > 0$ is arbitrary here, we have that $\|(x_1)(x_2)\| \leq \|(x_1)\| \|(x_2)\|$, as required.

Next, notice that $\|(e)\| \leq \|e\| = 1$. On the other hand, for all $x \in A$, we have that $\|(x)\| = \|(x)(e)\| \leq \|(x)\| \|(e)\|$, so $\|(e)\| \geq 1$. \[\Box\]

**Theorem 8.3.** (a) If $I \neq A$ is an ideal, then $I \cap G(A) = \emptyset$.
(b) The closure of an ideal is an ideal.
(c) Every maximal ideal is closed.
(d) Every ideal $I \neq A$ is contained in some maximal ideal $J \supset I$.
Proof. (a) If \( x \in I \cap G(A) \), then \( y = x(x^{-1}y) \in I \) for all \( y \in A \), so \( I = A \).

(b) The closure of a subspace is a subspace, and if \( x \in \overline{I} \), \( y \in A \), then there are \( x_n \in I \), \( x_n \to x \). Thus \( x_n y \in I \) and \( x_n y \to xy \) by the continuity of the multiplication, so \( xy \in \overline{I} \), as required.

(c) Let \( I \) be a maximal ideal. Then, by (b), \( \overline{I} \) is another ideal that contains \( I \). Since \( I \cap G(A) = \emptyset \), by (a), and since \( G(A) \) is open, \( \overline{I} \) still doesn’t intersect \( G(A) \). In particular, \( \overline{I} \neq A \), so \( \overline{I} = I \) because \( I \) was maximal.

(d) This follows in the usual way from Zorn’s Lemma. Also as usual, we don’t want to discuss the details of this argument here. \( \square \)

**Definition 8.4.** The *spectrum* or maximal ideal space \( \Delta \) of a commutative Banach algebra \( A \) is defined as

\[
\Delta = \{ \phi : A \to \mathbb{C} : \phi \text{ complex homomorphism} \}.
\]

The term *maximal ideal space* is justified by parts (a) and (b) of the following result, which set up a one-to-one correspondence between complex homomorphisms and maximal ideals.

**Theorem 8.5.** (a) If \( I \) is a maximal ideal, then there exists a unique \( \phi \in \Delta \) with \( N(\phi) = I \).

(b) Conversely, if \( \phi \in \Delta \), then \( N(\phi) \) is a maximal ideal.

(c) \( x \in G(A) \iff \phi(x) \neq 0 \) for all \( \phi \in \Delta \).

(d) \( x \in G(A) \iff x \) does not belong to any ideal \( I \neq A \).

(e) \( z \in \sigma(x) \iff \phi(x) = z \) for some \( \phi \in \Delta \).

Proof. (a) A maximal ideal is closed by Theorem 8.3(c), so the quotient \( A/I \) is a Banach algebra by Theorem 8.2. Let \( x \in A \), \( x \notin I \), and put \( J = \{ ax + y : a \in A, y \in I \} \). It’s easy to check that \( J \) is an ideal, and \( J \supset I \), because we can take \( a = 0 \). Moreover, \( x = ex + 0 \in J \), but \( x \notin I \), so, since \( I \) is maximal, we must have that \( J = A \). In particular, \( e \in J \), so there are \( a \in A \), \( y \in I \) so that \( ax + y = e \). Thus \( (a)(x) = (e) \) in \( A/I \). Since \( x \in A \) was an arbitrary vector with \( x \notin I \), we have shown that every \( (x) \in A/I \), \( (x) \neq 0 \) is invertible. By the Gelfand-Mazur Theorem, \( A/I \cong \mathbb{C} \). More precisely, there exists an isometric homomorphism \( f : A/I \to \mathbb{C} \). The map \( A \to A/I \), \( x \mapsto (x) \) also is a homomorphism (the algebraic structure on \( A/I \) is defined in such a way that this would be true), so the composition \( \phi(x) := f((x)) \) is another homomorphism: \( \phi \in \Delta \). Since \( f \) is injective, its kernel consists of exactly those \( x \in A \) that are sent to zero by the first homomorphism, that is, \( N(\phi) = I \).

It remains to establish uniqueness. If \( N(\phi) = N(\psi) \), then \( x - \psi(x)e \in N(\phi) \) for all \( x \in A \), so \( 0 = \phi(x) - \psi(x) \).
(b) Homomorphisms are continuous, so $N(\phi)$ is a closed linear subspace. If $x \in N(\phi)$, $y \in A$, then $\phi(xy) = \phi(x)\phi(y) = 0$, so $xy \in N(\phi)$ also, and $N(\phi)$ is an ideal. Since $\phi : A \to \mathbb{C}$ is a linear map to the one-dimensional space $\mathbb{C}$, we have that $\text{codim } N(\phi) = 1$, so $N(\phi)$ is already maximal as a subspace (the only strictly bigger subspace is $A$).

(c) $\implies$: This was proved earlier, in Proposition 7.3.

$\Leftarrow$: Suppose that $x \notin G(A)$. Then $I_0 = \{ax : a \in A\}$ is an ideal with $I_0 \neq A$ (because $e \notin I_0$). By Theorem 8.3(d), there exists a maximal ideal $I \supset I_0$. By part (a), there is a $\phi \in \Delta$ with $N(\phi) = I$. In particular, $\phi(x) = 0$.

(d) This follows immediately from what we have shown already, plus Theorem 8.3(d) again.

(e) We have $z \in \sigma(x)$ if and only if $x - ze \notin G(A)$, and by part (c), this holds if and only if $\phi(x - ze) = \phi(x) - z = 0$ for some $\phi \in \Delta$. □

In particular, this says that a commutative Banach algebra always admits complex homomorphisms, that is, we always have $\Delta \neq \emptyset$. Indeed, notice that Theorem 8.3(d) with $I = \{0\}$ shows that there are maximal ideals, so we obtain the claim from Theorem 8.5(a). Alternatively, we could use Theorem 8.5(e) together with the fact that spectra are always non-empty (Theorem 7.8(a)). The situation can be quite different on non-commutative algebras:

Exercise 8.1. Consider the algebra $\mathbb{C}^{2\times 2} = B(\mathbb{C}^2)$ of $2\times 2$-matrices (this becomes a Banach algebra if we fix an arbitrary norm on $\mathbb{C}^2$ and use the corresponding operator norm; however, as this is a purely algebraic exercise, the norm plays no role here). Show that there are no complex homomorphisms $\phi \neq 0$ on this algebra.

Here is a rather spectacular application of the ideas developed in Theorem 8.5:

Example 8.1. Consider the Banach algebra of absolutely convergent trigonometric series:

$$A = \left\{ f(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n e^{inx} : a \in \ell^1(\mathbb{Z}) \right\}$$

We have written $f(e^{ix})$ rather than $f(x)$ because it will be convenient to think of $f$ as a function on the unit circle $S = \{z \in \mathbb{C} : |z| = 1\} = \{e^{ix} : x \in \mathbb{R}\}$. Notice that the series converges uniformly, so $A \subset C(S)$.

Exercise 8.2. Show that if $f \equiv 0$, then $a_n = 0$ for all $n \in \mathbb{Z}$.

Suggestion: Recall that $\{e^{inx}\}$ is an ONB of $L^2((-\pi, \pi), dx/(2\pi))$. Use this fact to derive a formula that recovers the $a_n$’s from $f$. 
The algebraic operations on $A$ are defined pointwise; for example, $(f + g)(z) := f(z) + g(z)$. It is not entirely clear that the product of two functions from $A$ will be in $A$ again, but this issue will be addressed later.

Consider the map $\varphi : \ell^1 \to A$, $\varphi(a) = \sum a_n e^{inx}$. It is clear that $\varphi$ is linear and surjective. Moreover, Exercise 8.2 makes sure that $\varphi$ is injective. Therefore, we can define a norm on $A$ by $\|\varphi(a)\| = \|a\|_1$. This makes $A$ isometrically isomorphic to $\ell^1(\mathbb{Z})$ as a Banach space. We claim that these spaces are actually isometrically isomorphic as Banach algebras, where we endow $\ell^1$ with the convolution product, as in Example 7.5:

$$(a * b)_n = \sum_{j=-\infty}^{\infty} a_j b_{n-j}$$

Exercise 8.3. Show that $\varphi$ is a homomorphism. Since we already know that $\varphi$ is linear, you must show that $\varphi(a * b) = \varphi(a) \varphi(b)$.

In particular, this does confirm that $fg \in A$ if $f, g \in A$ (the sequence corresponding to $fg$ is $a * b$ if $a$ and $b$ correspond to $f$ and $g$, respectively). Since $\ell^1(\mathbb{Z})$ is a Banach algebra, $A$ is a Banach algebra also, or perhaps it would be more appropriate to say that $A$ is another realization of the same Banach algebra.

Proposition 8.6. Every $\varphi \in \Delta$ on this Banach algebra is an evaluation: There exists a $z \in S$ so that $\varphi(f) = f(z)$. Conversely, this formula defines a complex homomorphism for every $z = e^{it} \in S$.

Exercise 8.4. Prove Proposition 8.6, by using the following strategy: Let $\varphi \in \Delta$. What can you say about $|\varphi(e^{ix})|$ and $|\varphi(e^{-ix})|$? Conclude that $|\varphi(e^{ix})| = 1$, say $\varphi(e^{ix}) = e^{it}$. Now use the continuity of $\varphi$ to prove that for an arbitrary $f \in A$, we have that $\varphi(f) = f(e^{it})$.

The converse is much easier, of course.

This material leads to an amazingly elegant proof of the following result:

Theorem 8.7 (Wiener). Consider an absolutely convergent trigonometric series: $f(e^{ix}) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $a \in \ell^1(\mathbb{Z})$. Suppose that $f(z) \neq 0$ for all $z \in S$. Then $1/f$ also has an absolutely convergent trigonometric expansion: There exists $b \in \ell^1(\mathbb{Z})$ so that $1/f(e^{ix}) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$.

This result is interesting because it is usually very hard to tell whether the expansion coefficients (“Fourier coefficients”) of a given function lie in $\ell^1$. 
Proof. By Proposition 8.6, the hypothesis says that \( \phi(f) \neq 0 \) for all \( \phi \in \Delta \). By Theorem 8.5(c), \( f \in G(A) \). Clearly, the inverse is given by the function \( 1/f \).

We now come to the most important topic of this chapter. With each \( x \in A \), we can associate a function \( \hat{x} : \Delta \to \mathbb{C} \), \( \hat{x}(\phi) = \phi(x) \). We have encountered this type of construction before (see Proposition 4.3); it will work especially well in this new context. We call \( \hat{x} \) the Gelfand transform of \( x \). The Gelfand topology on \( \Delta \subset A^* \) is defined as the relative topology that is induced by the weak-* topology on \( A^* \). By Exercise 4.10, this is also the weak topology that is generated by the maps \( \{ \hat{x} : A \to \mathbb{C} : x \in A \} \). We also write \( \hat{A} \) for this collection of maps.

Here are the fundamental properties of the Gelfand transform.

**Theorem 8.8.** (a) \( \Delta \) with the Gelfand topology is a compact Hausdorff space.
(b) \( \hat{A} \subset C(\Delta) \) and the Gelfand transform \( \hat{\cdot} : A \to C(\Delta) \) is a homomorphism between Banach algebras.
(c) \( \sigma(x) = \hat{x}(\Delta) = \{ \hat{x}(\phi) : \phi \in \Delta \} \); in particular, \( \| \hat{x} \|_\infty = r(x) \leq \| x \| \).

Note that we use the term Gelfand transform for the function \( \hat{x} \in C(\Delta) \), but also for the homomorphism \( \hat{\cdot} : A \to C(\Delta) \) that sends \( x \) to \( \hat{x} \). Recall from Proposition 7.9 that in the Banach algebra \( C(\Delta) \), \( \sigma(\hat{x}) = \hat{x}(\Delta) \), so part (c) of the Theorem really says that the Gelfand transform preserves spectra: \( \sigma(\hat{x}) = \sigma(x) \). It also preserves the algebraic structure (by part (b)) and is continuous (by part (c) again).

**Proof.** (a) This is very similar to the proof of the Banach-Alaoglu Theorem, so we will just provide a sketch. From that result, we know that \( \Delta \subset \overline{B}_1(0) = \{ F \in A^* : \| F \| \leq 1 \} \) is a subset of the compact Hausdorff space \( \overline{B}_1(0) \), and so it again suffices to show that \( \Delta \) is closed in the weak-* topology. A procedure very similar to the one used in the original proof works again: If \( \psi \in \overline{B}_1(0) \setminus \Delta \), then either \( \psi \equiv 0 \) or there exist \( x, y \in A \) so that \( \epsilon := |\psi(xy) - \psi(x)\psi(y)| > 0 \). Let us indicate how to finish the proof in the second case: Let

\[
U = \left\{ \phi \in \overline{B}_1(0) : |\phi(xy) - \psi(xy)| < \frac{\epsilon}{3}, |\psi(x)| |\phi(y) - \psi(y)| < \frac{\epsilon}{3}, |\phi(y)| < |\psi(y)| + 1, (|\psi(y)| + 1) |\phi(x) - \psi(x)| < \frac{\epsilon}{3} \right\}.
\]
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Then $U$ is an open set in the weak-$*$ topology that contains $\psi$. Moreover, if $\phi \in U$, then

$$|\phi(xy) - \phi(x)\phi(y)| \geq |\psi(xy) - \psi(x)\psi(y)| - |\phi(xy) - \psi(xy)| - |\phi(y)||\phi(x) - \psi(x)| - |\psi(x)||\phi(y) - \psi(y)| > \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} = 0,$$

so $\phi \not\in \Delta$ either and indeed $\Delta \cap U = \emptyset$. We have shown that $\overline{B}_1(0) \setminus \Delta$ is open, as claimed.

(b) It is clear that $\hat{A} \subset C(\Delta)$, from the second description of the Gelfand topology as the weakest topology that makes all maps $\hat{x} \in \hat{A}$ continuous. To prove that $\hat{\cdot}: A \to C(\Delta)$ is a homomorphism of algebras, we compute

$$(xy)\hat{\cdot}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}(\phi)\hat{y}(\phi) = (\hat{x}\hat{y}) (\phi);$$

in other words, $(xy)\hat{\cdot} = \hat{x}\hat{y}$. Similar arguments show that $\hat{\cdot}$ is also linear.

(c) This is an immediate consequence of Theorem 8.5(e). \qed

Let us summarize this one more time and also explore the limitations of the Gelfand transform. The maximal ideal space $\Delta$ with the Gelfand topology is a compact Hausdorff space, and the Gelfand transform provides a map from the original (commutative!) Banach algebra $A$ to $C(\Delta)$ that

- preserves the algebraic structure: it is a homomorphism;
- preserves spectra: $\sigma(\hat{x}) = \sigma(x)$;
- is continuous: $\|\hat{x}\| \leq \|x\|$.

However, in general, it

- does not preserve the norm: it need not be isometric; in fact, it can have a non-trivial null space;
- need not be surjective; worse still, its range $\hat{A}$ need not be a closed subspace of $C(\Delta)$.

Another remarkable feature of the Gelfand transform is the fact that it is a purely algebraic construction: it is independent of the norm that is being used on $A$. Indeed, all we need to do is construct the complex homomorphisms on $A$ and then evaluate these on $x$ to find $\hat{x}$. We also let the $\hat{x}$ generate a weak topology on $\Delta$, but again, if formulated this way, this procedure does not involve the norm on $A$.

We are using the fact that there is some norm on $A$, though, for example to make sure that $\Delta$ is a compact space in the Gelfand topology. However, the Gelfand transform does not change if we switch to
a different norm on $A$ (in many situations, there will be only one norm that makes $A$ a Banach algebra).

The following examples illustrate the last two properties from the above list.

**Example 8.2.** Let $A$ be the set of matrices of the form $T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. This is a commutative Banach algebra if we use matrix multiplication and an arbitrary operator norm on $A$; in fact, $A$ is a (commutative) subalgebra of $\mathbb{C}^{2 \times 2} = B(\mathbb{C}^2)$.

**Exercise 8.5.** Find all complex homomorphisms. Then show that there are $T \in A$, $T \neq 0$ with $\phi(T) = 0$ for all $\phi \in \Delta$. In other words, $\widehat{T} = 0$, so the Gelfand transform on $A$ is not injective.

**Remark:** To get this started, you could use the fact that homomorphisms are in particular linear functionals, and we know what these are on a finite-dimensional vector space.

**Example 8.3.** We consider again the Banach algebra of absolutely convergent trigonometric series from Example 8.1. We saw in Proposition 8.6 that as a set, $\Delta$ may be identified with the unit circle $S = \{z : |z| = 1\}$. To extract this identification from Proposition 8.6, notice also that if $z, z' \in S$, $z \neq z'$, then there will be an $f \in A$ with $f(z) \neq f(z')$. Actually, there will be a trigonometric polynomial (that is, $a_n = 0$ for all large $|n|$) with this property. So if $z \neq z'$, then the corresponding homomorphisms are also distinct.

With this identification of $\Delta$ with $S$, the Gelfand transform $\widehat{f}$ of an $f \in A$ is the function that sends $z \in S$ to $\phi_z(f) = f(z)$; in other words, $\widehat{f}$ is just $f$ itself. The Gelfand topology on $S$ is the weakest topology that makes all $\widehat{f}$ continuous. Clearly, these functions are continuous if we use the usual topology on $S$. Moreover, $S$ with both topologies is a compact Hausdorff space. Now the following simple but very important Lemma shows that the Gelfand topology is just the usual topology on $S$.

**Lemma 8.9.** Let $\mathcal{T}_1 \subset \mathcal{T}_2$ be topologies on a common space $X$. If $X$ is a compact Hausdorff space with respect to both topologies, then $\mathcal{T}_1 = \mathcal{T}_2$.

**Proof.** We use the fact that on a compact Hausdorff space, a subset is compact if and only if it is closed. Now let $U \in \mathcal{T}_2$. Then $U^c$ is closed in $\mathcal{T}_2$, thus compact. But then $U^c$ is also compact with respect to $\mathcal{T}_1$, because $\mathcal{T}_1$ is a weaker topology (there are fewer open covers to consider). Thus $U^c$ is $\mathcal{T}_1$-closed, so $U \in \mathcal{T}_1$. \qed

$\widehat{A} = A$ is dense in $C(S) = C(\Delta)$ because, by (a suitable version of) the Weierstraß approximation theorem, every continuous function
Commutative Banach algebras on $S$ can be uniformly (that is, with respect to $\| \cdot \|_{\infty}$) approximated by trigonometric polynomials, and these manifestly are in $A$. However, $A \neq C(S)$. This is a well known fact from the theory of Fourier series. The following Exercise outlines an argument of a functional analytic flavor.

Exercise 8.6. Suppose that we had $\widehat{A} = C(S)$. First of all, use Corollary 3.3 to show that then
\begin{equation}
\|a\|_1 \leq C\|f\|_{\infty}
\end{equation}
for all $a \in \ell^1$ and $f(x) = \sum a_n e^{inx}$, for some $C > 0$.

However, (8.1) can be refuted by considering approximations $f_N$ to the (discontinuous!) function $f(e^{ix}) = \chi_{(0,\pi)}(x)$. More precisely, proceed as follows: Notice that if $f = \sum a_n e^{inx}$ with $a \in \ell^1$, then the series also converges in $L^2(-\pi,\pi)$. Recall that $\{e^{inx}\}$ is an ONS (in fact, an ONB) in $L^2((-\pi,\pi), dx/(2\pi))$, so it follows that
\begin{equation*}
a_n = \langle e^{inx}, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{ix}) e^{-inx} \, dx
\end{equation*}
for all $f \in C(S)$. Use this to approximately compute the $a_n^{(N)}$ for functions $f_N \in C(S)$ that satisfy $0 \leq f_N \leq 1$, $f_N(e^{ix}) = 1$ for $0 < x < \pi$ and $f_N(e^{ix}) = 0$ for $-\pi + 1/N < x < -1/N$. Show that $\|a^{(N)}\|_1$ can be made arbitrarily large by taking $N$ large enough. Since $\|f_N\|_{\infty} = 1$, this contradicts (8.1).

Exercise 8.7. (a) Show that $c$ with pointwise multiplication is a Banach algebra.
(b) Show that $\ell^1 \subset c$ is an ideal.
(c) Show that there is a unique maximal ideal $I \supset \ell^1$. Find $I$ and also the unique $\phi \in \Delta$ with $N(\phi) = I$.

Exercise 8.8. Consider the Banach algebra $\ell^\infty$. Show that
\[ I_n = \{ x \in \ell^\infty : x_n = 0 \} \]
is a maximal ideal for every $n \in \mathbb{N}$. Find the corresponding homomorphisms $\phi_n \in \Delta$ with $N(\phi_n) = I_n$. Finally, show that there must be additional complex homomorphisms (Suggestion: Find another ideal $J$ that is not contained in any $I_n$.)

Exercise 8.9. Let $A$ be a commutative Banach algebra. Show that the spectral radius satisfies
\[ r(xy) \leq r(x)r(y), \quad r(x + y) \leq r(x) + r(y) \]
for all $x, y \in A$. 

Exercise 8.10. Show that the inequalities from Exercise 8.9 can fail on non-commutative Banach algebras. More specifically, show that they fail on $A = \mathbb{C}^{2 \times 2}$.

Remark: Recall that on this Banach algebra, the spectrum of a matrix is the set of its eigenvalues, so $r(T)$ is the absolute value of the biggest eigenvalue of $T$. 