6. Operators in Hilbert spaces

Let H be a Hilbert space. In this chapter, we are interested in basic properties of operators $T \in B(H)$ on this Hilbert space. First of all, we would like to define an adjoint operator T^*, and its defining property should be given by $\langle T^*y, x\rangle = \langle y, Tx\rangle$. It is not completely clear, however, that this indeed defines a new operator T^*. To make this idea precise, we proceed as follows: Fix $y \in H$ and consider the map $H \to \mathbb{C}$, $x \mapsto \langle y, Tx\rangle$. It is clear that this is a linear map, and

$$|\langle y, Tx\rangle| \le \|y\| \, \|Tx\| \le \|y\| \, \|T\| \, \|x\|,$$

so the map is also bounded. By the Riesz Representation Theorem, there exists a unique vector $z = z_y \in H$ with $\langle y, Tx\rangle = \langle z_y, x\rangle$ for all $x \in H$. We can now define a map $T^* : H \to H$, $T^*y = z_y$. By construction, we then indeed have $\langle T^*y, x\rangle = \langle y, Tx\rangle$ for all $x, y \in H$; conversely, this condition uniquely determines T^*y for all $y \in H$. We call T^* the *adjoint operator* (of T).

Theorem 6.1. *Let $S, T \in B(H)$, $c \in \mathbb{C}$. Then:*
(a) $T^* \in B(H)$;
(b) $(S+T)^* = S^* + T^*$, $(cT)^* = \overline{c}T^*$;
(c) $(ST)^* = T^*S^*$;
(d) $T^{**} = T$;
(e) If T is invertible, then T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$;
(f) $\|T\| = \|T^*\|$, $\|TT^*\| = \|T^*T\| = \|T\|^2$ (the C^* property*)*

Here, we call $T \in B(H)$ *invertible* (it would be more precise to say: invertible in $B(H)$) if there exists an $S \in B(H)$ with $ST = TS = 1$. In this case, S with these properties is unique and we call it the inverse of T and write $S = T^{-1}$. Notice that this version of invertibility requires more than just injectivity of T as a map: we also require the inverse map to be continuous and defined everywhere on H (and linear, but this is automatic). So we can also say that $T \in B(H)$ is invertible (in this sense) precisely if T is bijective on H and has a continuous inverse. Actually, Corollary 3.3 shows that this continuity is automatic also, so $T \in B(H)$ is invertible precisely if T is a bijective map.

Exercise 6.1. (a) Show that it is not enough to have just one of the equations $ST = 1$, $TS = 1$: Construct two non-invertible maps $S, T \in B(H)$ (on some Hilbert space H; $H = \ell^2$ seems a good choice) that nevertheless satisfy $ST = 1$.
(b) However, if H is finite-dimensional and $ST = 1$, then both S and T will be invertible. Prove this.

Proof. (a) The (anti-)linearity of the scalar product implies that T^* is linear; for example, $\langle cT^*y, x\rangle = \langle T^*y, \overline{c}x\rangle = \langle y, T(\overline{c}x)\rangle = \langle cy, Tx\rangle$ for all $x \in H$, so $T^*(cy) = cT^*y$. Furthermore,

$$\sup_{\|y\|=1} \|T^*y\| = \sup_{\|x\|=\|y\|=1} |\langle T^*y, x\rangle| = \sup_{\|x\|=\|y\|=1} |\langle y, Tx\rangle| = \|T\|,$$

so $T^* \in B(H)$ and $\|T^*\| = \|T\|$.

Parts (b), (c) follow directly from the definition of the adjoint operator. For example, to verify (c), we just observe that $\langle y, STx\rangle = \langle S^*y, Tx\rangle = \langle T^*S^*y, x\rangle$, so $(ST)^*y = T^*S^*y$.

(d) We have $\langle y, T^*x\rangle = \overline{\langle T^*x, y\rangle} = \overline{\langle x, Ty\rangle} = \langle Ty, x\rangle$, so $T^{**}y = Ty$.

(e) Obviously, $1^* = 1$. So if we take adjoints in $TT^{-1} = T^{-1}T = 1$ and use (c), we obtain $(T^{-1})^*T^* = T^*(T^{-1})^* = 1$. Since $(T^{-1})^* \in B(H)$, this says that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

(f) We already saw in the proof of part (a) that $\|T^*\| = \|T\|$. It is then also clear that $\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2$. On the other hand,

$$\|T^*T\| = \sup_{\|x\|=\|y\|=1} |\langle y, T^*Tx\rangle| \ge \sup_{\|x\|=1} |\langle x, T^*Tx\rangle| = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2,$$

so $\|T^*T\| = \|T\|^2$. If applied to T^* in place of T, this also shows that $\|TT^*\| = \|T^{**}T^*\| = \|T^*\|^2 = \|T\|^2$. □

Theorem 6.2. *Let $T \in B(H)$. Then $N(T^*) = R(T)^\perp$.*

Proof. We have $x \in N(T^*)$ precisely if $\langle T^*x, y\rangle = 0$ for all $y \in H$, and this happens if and only if $\langle x, Ty\rangle = 0$ $(y \in H)$. This, in turn, holds if and only if $x \in R(T)^\perp$. □

We will be especially interested in Hilbert space operators with additional properties.

Definition 6.3. Let $T \in B(H)$. We call T *self-adjoint* if $T = T^*$, *unitary* if $TT^* = T^*T = 1$ and *normal* if $TT^* = T^*T$.

So self-adjoint and unitary operators are also normal. We introduced unitary operators earlier, in Chapter 5, in the more general setting of operators between two Hilbert spaces; recall that we originally defined these as maps that preserve the complete Hilbert space structure (that is, the algebraic structure and the scalar product). Theorem 6.4(b) below will make it clear that the new definition is equivalent to the old one (for maps on one space). Also, notice that U is unitary precisely if U is invertible (in $B(H)$, as above) and $U^{-1} = U^*$.

Here are some additional reformulations:

Theorem 6.4. *Let $U \in B(H)$. Then the following statements are equivalent:*

(a) U is unitary;
(b) U is bijective and $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in H$;
(c) U is surjective and isometric (that is, $\|Ux\| = \|x\|$ for all $x \in H$).

Exercise 6.2. Prove Theorem 6.4. *Suggestion:* Use polarization to derive (b) from (c).

We now take a second look at (orthogonal) projections. Recall that, by definition, the projection on the closed subspace $M \subseteq H$ is the operator that sends $x \in H$ to $y \in M$, where y is the part from M in the (unique) decomposition $x = y + z$, $y \in M$, $z \in M^\perp$. If $P = P_M$ is such a projection, then it has the following properties: $P^2 = P$ (see Proposition 5.10), $R(P) = M$, $N(P) = M^\perp$

Exercise 6.3. Prove these latter two properties. Also, show that $Px = x$ if and only if $x \in M = R(P)$.

Theorem 6.5. *Let $P \in B(H)$. Then the following are equivalent:*
(a) P is a projection;
(b) $1 - P$ is a projection;
(c) $P^2 = P$ and $R(P) = N(P)^\perp$;
(d) $P^2 = P$ and P is self-adjoint;
(e) $P^2 = P$ and P is normal.

Proof. (a) $\Longrightarrow$ (b): It is clear from the definition of P_M and the fact that $M^{\perp\perp} = M$ that $1 - P$ is the projection onto $M^\perp$ if P is the projection onto M.

(b) $\Longrightarrow$ (a): This is the same statement, applied to $1 - P$ in place of P.

(a) $\Longrightarrow$ (c): This was already observed above, see Exercise 6.3.

(c) $\Longrightarrow$ (a): If $y \in R(P)$, so $y = Pu$ for some $u \in H$, then $Py = P^2u = Pu = y$. On the other hand, if $z \in R(P)^\perp = N(P)$ (and here we make use of the fact that $N(P)$ is a closed subspace, because P is continuous), then $Pz = 0$. Now let $x \in H$ be arbitrary and use Theorem 5.8 to decompose $x = y + z$, $y \in R(P)$, $z \in R(P)^\perp$. Note that $R(P)$ is a closed subspace because it is the orthogonal complement of $N(P)$ by assumption. By our earlier observations, $Px = Py + Pz = y$, so indeed P is the projection on $R(P)$.

(a) $\Longrightarrow$ (d): Again, we already know that $P^2 = P$. Moreover, for arbitrary $x, y \in H$, we have

$$\langle Px, Py\rangle = \langle x, Py\rangle = \langle Px, y\rangle, \tag{6.1}$$

because, for example, $x = Px + (1 - P)x$, but $(1 - P)x \perp Py$. The second equality in (6.1) says that $P^* = P$, as desired.

(d) $\Longrightarrow$ (e) is trivial.

(e) $\Longrightarrow$ (c): Since P is normal, we have

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PP^*x, x\rangle = \|P^*x\|^2.$$

In particular, this implies that $N(P) = N(P^*)$, and Theorem 6.2 then shows that $N(P) = R(P)^\perp$. We could finish the proof by passing to the orthogonal complements here if we also knew that $R(P)$ is closed. We will establish this by showing that $R(P) = N(1-P)$ (which is closed, being the null space of a continuous operator). Clearly, if $x \in R(P)$, then $x = Py$ for some $y \in H$ and thus $(1-P)x = P^2y - Py = 0$, so $x \in N(1-P)$. Conversely, if $x \in N(1-P)$, then $x = Px \in R(P)$. $\square$

For later use, we also note the following technical property of projections:

Proposition 6.6. *Let P, Q be projections. Then PQ is a projection if and only if $PQ = QP$. In this case, $R(PQ) = R(P) \cap R(Q)$.*

Proof. If PQ is a projection, then it satisfies condition (d) from Theorem 6.5, so $PQ = (PQ)^* = Q^*P^* = QP$. Conversely, if we assume that $PQ = QP$, then the same calculation shows that PQ is self-adjoint. Moreover, $(PQ)^2 = PQPQ = PPQQ = P^2Q^2 = PQ$, and now Theorem 6.5 shows that PQ is a projection.

To find its range, we observe that $R(PQ) \subseteq R(P)$, but also $R(PQ) = R(QP) \subseteq R(Q)$, so $R(PQ) \subseteq R(P) \cap R(Q)$. On the other hand, if $x \in R(P) \cap R(Q)$, then $Px = Qx = x$, so $PQx = x$ and thus $x \in R(PQ)$. $\square$

On the finite-dimensional Hilbert space $H = \mathbb{C}^n$, every operator $T \in B(\mathbb{C}^n)$ (equivalently, every matrix $T \in \mathbb{C}^{n\times n}$) can be brought to a relatively simple canonical form (the Jordan normal form) by a change of basis. In fact, usually operators are diagonalizable.

Exercise 6.4. Can you establish the following precise version: The set of diagonalizable matrices contains a dense open subset of $\mathbb{C}^{n\times n}$; here we use the topology generated by the operator norm. (In fact, by Theorem 2.15, any other norm will give the same topology.)

The situation on infinite-dimensional Hilbert spaces is much more complicated. We cannot hope for a normal form theory for *general* Hilbert space operators. In fact, the following much more modest question is a famous long-standing open problem:

Does every $T \in B(H)$ have a non-trivial invariant subspace?
(the invariant subspace problem)

Here, a closed subspace $M \subseteq H$ is called invariant if $TM \subseteq M$; the trivial invariant subspaces are $\{0\}$ and H.

Exercise 6.5. (a) Show that every $T \in \mathbb{C}^{n\times n} = B(\mathbb{C}^n)$ has a non-trivial invariant subspace.
(b) Show that $\overline{L(\{T^n x : n \geq 0\})}$ is an invariant subspace (possibly trivial) for every $x \in H$.
(c) Deduce from (b) that every $T \in B(H)$ on a *non-separable* Hilbert space H has a non-trivial invariant subspace.

Of course, we wouldn't really gain very much even from a positive answer to the invariant subspace problem; this would just make sure that every operator has some smaller part that could be considered separately. The fact that the invariant subspace problem is universally recognized as an exceedingly hard problem makes any attempt at a general structure theory for Hilbert space operators completely hopeless.

We will therefore focus on normal operators, which form an especially important subclass of Hilbert space operators. Here, we will be able to develop a powerful theory. The fundamental result here is the *spectral theorem;* we will prove this in Chapter 10, after a few detours. It is also useful to recall from linear algebra that a normal *matrix* $T \in \mathbb{C}^{n\times n}$ can be diagonalized; in fact, this is done by changing from the original basis to a new ONB, consisting of the eigenvectors of T.

Generally speaking, the eigenvalues and eigenvectors of a matrix take center stage in the analysis in the finite-dimensional setting, so it seems a good idea to try to generalize these notions. We do this as follows (actually, we only generalize the concept of an eigenvalue here):

Definition 6.7. For $T \in B(H)$, define

$$\rho(T) = \{z \in \mathbb{C} : T - z \text{ is invertible in } B(H)\},$$
$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

We call $\rho(T)$ the *resolvent set* of T and $\sigma(T)$ the *spectrum* of T.

Exercise 6.6. Show that $\sigma(T)$ is exactly the set of eigenvalues of T if $T \in B(\mathbb{C}^n)$ is a matrix.

This confirms that we may hope to have made a good definition, but perhaps the more obvious try would actually have gone as follows: Call $z \in \mathbb{C}$ an *eigenvalue* of $T \in B(H)$ if there exists an $x \in H$, $x \neq 0$, such that $Hx = zx$, and introduce $\sigma_p(T)$ as the set of eigenvalues of T; we also call $\sigma_p(T)$ the *point spectrum* of T.

However, this doesn't work very well in the infinite-dimensional setting:

Exercise 6.7. Consider the operator $S \in \ell^2(\mathbb{Z})$, $(Sx)_n = x_{n+1}$ (S as in shift), and prove the following facts about S:
(a) S is unitary;
(b) $\sigma_p(S) = \emptyset$.

We can also obtain an example of a self-adjoint operator with no eigenvalues from this, by letting $T = S + S^*$. Then $T = T^*$ (obvious), and again $\sigma_p(T) = \emptyset$ (not obvious, and in fact you will probably need to use a few facts about difference equations to prove this; this part of the problem is optional).

Exercise 6.8. Show that $\sigma_p \subseteq \sigma$.

Exercise 6.9. Here's another self-adjoint operator with no eigenvalues; compare Exercise 6.7. Define $T : L^2(0,1) \to L^2(0,1)$ by $(Tf)(x) = xf(x)$.
(a) Show that $T \in B(L^2(0,1))$ and $T = T^*$, and compute $\|T\|$.
(b) Show that $\sigma_p(T) = \emptyset$. Can you also show that $\sigma(T) = [0,1]$?

Exercise 6.10. Let $s(x,y)$ be a sesquilinear form that is bounded in the sense that

$$M \equiv \sup_{\|x\|=\|y\|=1} |s(x,y)| < \infty.$$

Show that there is a unique operator $T \in B(H)$ such that $s(x,y) = \langle x, Ty\rangle$. Show also that $\|T\| = M$.

Hint: Apply the Riesz Representation Theorem to the map $x \mapsto \overline{s(x,y)}$, for fixed but arbitrary $y \in H$.

Exercise 6.11. Let $T : H \to H$ be a linear operator, and assume that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Show that T is bounded (the *Hellinger-Toeplitz Theorem*).
Suggestion: Show that T is closed and apply the closed graph theorem.