15. Perturbations by compact operators

In this chapter, we study the stability (or lack thereof) of various spectral properties under small perturbations. Here's the type of situation we have in mind: Let $T \in B(H)$ be a self-adjoint operator, and let $V \in B(H)$ be another self-adjoint operator that will be assumed to be small in a suitable sense. We then want to compare the spectral properties of $T + V$ with those of T.

Definition 15.1. Let T be a self-adjoint operator, with spectral resolution E. The *essential spectrum* $\sigma_{ess}(T)$ is the set of all $t \in \mathbb{R}$ for which $\dim R(E((t-r, t+r))) = \infty$ for all $r > 0$.

Recall that if $t \notin \sigma(T)$, then $E((t-r, t+r)) = 0$ for all small $r > 0$, so $\sigma_{ess} \subset \sigma$. Also, it is clear that σ_{ess} is a closed subset of $\mathbb{R}$ because if $t \notin \sigma_{ess}$, then $R(E((t-r, t+r)))$ is finite-dimensional for some $r > 0$, but this implies that $(t-r, t+r) \cap \sigma_{ess} = \emptyset$, so the complement of σ_{ess} is open, as claimed.

Proposition 15.2. *$t \in \sigma_{ess}$ precisely if t is an accumulation point of σ or an eigenvalue of infinite multiplicity.*

Here, we define the multiplicity of an eigenvalue t as $\dim N(T - t)$, as expected. Of course, if T has finite spectral multiplicity, then the second alternative cannot occur, so in this case, σ_{ess} is just the set of accumulation points of σ. For example, this remark applies to Jacobi matrices.

Proof. If $t \in \sigma$ is not an accumulation point of the spectrum, then t is an isolated point of σ. So, for small enough $r > 0$, $E((t-r, t+r)) = E(\{t\})$. Since this is the projection onto $N(T - t)$, it will be finite-dimensional if t is not an eigenvalue of infinite multiplicity. Hence $t \notin \sigma_{ess}$.

Conversely, if t is an eigenvalue of infinite multiplicity, then $R(E(\{t\}) = N(T - t)$ is infinite-dimensional, so $t \in \sigma_{ess}$. If t is an accumulation point of σ, then, for any $r > 0$ and $N \in \mathbb{N}$, $(t-r, t+r)$ contains N distinct points $t_n \in \sigma$ and thus also N disjoint open subsets I_n that all intersect σ (just take small neighborhoods of the t_n's). Now $E(I_n) \neq 0$, so $\dim R(E(I_n)) \geq 1$, and, moreover, these subspaces are mutually orthogonal. Therefore, $\dim R(E((t-r, t+r))) \geq N$. Since N was arbitrary here, this space is in fact infinite-dimensional, so $t \in \sigma_{ess}$. $\square$

Exercise 15.1. Let $T \in B(H)$ be a self-adjoint operator on an infinite-dimensional Hilbert space H. Show that then $\sigma_{ess}(T) \neq \emptyset$.

It is sometimes also convenient to introduce a symbol for the complement, $\sigma_d = \sigma \setminus \sigma_{ess}$. We call σ_d the *discrete spectrum;* it consists of the isolated points of the spectrum (these are automatically eigenvalues) of finite multiplicity.

Here is our first result on perturbations.

Theorem 15.3 (Weyl). *Let T be a self-adjoint operator, and assume that V is compact and self-adjoint. Then $\sigma_{ess}(T+V) = \sigma_{ess}(T)$.*

There is a very useful criterion for a point to lie in the essential spectrum, which will lead to an effortless proof of Weyl's Theorem. We call a $x_n \in H$ a *Weyl sequence* (for T and t) if $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$, and $(T-t)x_n \to 0$.

Theorem 15.4. *$t \in \sigma_{ess}(T)$ if and only if there exists a Weyl sequence for T and t.*

It is tempting to compare this with the result of Exercise 10.20: $t \in \sigma(T)$ if and only there exists a sequence $x_n \in H$, $\|x_n\| = 1$, so that $(T-t)x_n \to 0$.

Proof. If $t \in \sigma_{ess}$, pick $x_1 \in R(E((t-1, t+1)))$, then $x_2 \in R(E((t-1/2, t+1/2)))$ with $x_2 \perp x_1$, then $x_3 \in R(E((t-1/3, t+1/3)))$ with $x_3 \perp x_1, x_2$ etc. We can also insist that $\|x_n\| = 1$. Then this procedure yields an ONS x_n, so $x_n \xrightarrow{w} 0$, and $\|(T-t)x_n\| \le 1/n$.

Conversely, assume that a Weyl sequence x_n has been constructed. We will argue by contradiction, so assume also that $\dim R(E((t-r, t+r))) < \infty$ for some $r > 0$. We abbreviate $P = E((t-r, t+r))$. Since $R(P)$ is finite-dimensional, P is a compact operator, and we assumed that $x_n \xrightarrow{w} 0$, so it follows that $\|Px_n\| \to 0$. Therefore,

$$\begin{aligned} \|(T-t)x_n\| &\ge \|(T-t)(1-P)x_n\| - \|(T-t)Px_n\| \\ &\ge r\|(1-P)x_n\| - \|(T-t)Px_n\| \to r, \end{aligned}$$

but this contradicts our assumption that x_n is a Weyl sequence. We have to admit that $t \in \sigma_{ess}$. □

Proof of Theorem 15.3. This is very easy now. If $x_n \xrightarrow{w} 0$, then $Vx_n \to 0$ by Theorem 14.6(b), so T and $T+V$ have the same Weyl sequences. □

Here are some typical applications of this result to Jacobi matrices.

Theorem 15.5. *Let J be a Jacobi matrix whose coefficients satisfy $a_n \to 1$, $b_n \to 0$. Then $\sigma_{ess}(J) = [-2, 2]$.*

Proof. Let J_0 be the Jacobi matrix with coefficients $a_n = 1$, $b_n = 0$. We know that $\sigma(J_0) = \sigma_{ess}(J_0) = [-2, 2]$. Now $J = J_0 + K$, where

$$(Ku)_n = (a_n - 1)u_{n+1} + (a_{n-1} - 1)u_{n-1} + b_n u_n \qquad (n \ge 2).$$

Exercise 15.2. Show that K is compact.
Suggestion: Show that we can write $K = K_0 + K_1$, where K_0 is a finite rank operator and $\|K_1\| < \epsilon$.

Now Weyl's Theorem gives the claim. □

The same argument shows that if, more generally, J, J' are Jacobi matrices whose coefficients satisfy $a_n - a'_n \to 0$, $b_n - b'_n \to 0$, then $\sigma_{ess}(J) = \sigma_{ess}(J')$. In particular, the essential spectrum only depends on what happens asymptotically, "at infinity."

We also obtain a decomposition theorem for whole line problems. By this, we mean the following: Consider a whole line Jacobi matrix $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, and let $J_\pm$ be its half line restrictions. More precisely, let

$$(J_+u)_n = \begin{cases} a_1u_2 + b_1u_1 & n = 1 \\ a_nu_{n+1} + a_{n-1}u_{n-1} + b_nu_n & n \geq 2 \end{cases},$$

$$(J_-u)_n = \begin{cases} a_{-1}u_{-1} + b_0u_0 & n = 0 \\ a_nu_{n+1} + a_{n-1}u_{n-1} + b_nu_n & n \leq -1 \end{cases}.$$

We interpret $J_\pm$ as an operator on $\ell^2(\mathbb{Z}_\pm)$, where $Z_+ = \mathbb{N}$, $Z_- = \mathbb{Z} \setminus \mathbb{N}$.

Theorem 15.6. $\sigma_{ess}(J) = \sigma_{ess}(J_+) \cup \sigma_{ess}(J_-)$

Proof. We will describe the argument somewhat informally, rather than try to set up elaborate notation for what is a fairly simple argument. We cut $\mathbb{Z}$ into two half lines and set $a_0 = 0$, which is a rank two perturbation of J, and thus preserves the essential spectrum by Weyl's Theorem. Call this new operator J_1. Since $\ell^2(\mathbb{Z}_+)$ is a reducing subspace for J_1, we may naturally identify $J_1 = J_+ \oplus J_-$. Therefore, the following observation finishes the proof.

Exercise 15.3. Let $T_j \in B(H_j)$ $(j = 1, 2)$ be self-adjoint operators, and let $T = T_1 \oplus T_2$. Show that then $\sigma_{ess}(T) = \sigma_{ess}(T_1) \cup \sigma_{ess}(T_2)$.

□

Theorem 15.4 is also often useful as a tool to investigate σ_{ess}. As an illustration, we will now discuss such an application. We need some notation. For simplicity, we only treat one-dimensional Schrödinger operators here; however, analogous results could be formulated and proved for Jacobi matrices also. Let $W \in \ell^\infty(\mathbb{Z})$, and denote the corresponding Schrödinger operator on $\ell^2(\mathbb{Z})$ by H_W. In other words, $(H_Wu)_n = u_{n+1} + u_{n-1} + W_nu_n$. Suppose that $V \in \ell^\infty(\mathbb{N})$ contains arbitrarily large chunks of W, in the following sense. There are numbers

$c_n, L_n \in \mathbb{N}$, $L_n \to \infty$, so that the sets $\{c_n - L_n, \ldots, c_n + L_n\}$ are disjoint subintervals of $\mathbb{N}$, and

$$V_{c_n+j} = W_j \qquad (|j| \leq L_n).$$

We denote the corresponding Schrödinger operator by H_V^+. The superscript $+$ reminds us that this is a half line operator, on $\ell^2(\mathbb{N})$.

Theorem 15.7. $\sigma(H_W) \subset \sigma_{ess}(H_V^+)$

Proof. Let $t \in \sigma(H_W)$. We will construct a Weyl sequence for H_V^+ and this t; this will finish the proof by Theorem 15.4.

By Exercise 10.20, there is a sequence $u^{(n)} \in \ell^2(\mathbb{Z})$ so that $\|u^{(n)}\| = 1$, $\|(H_W - t)u^{(n)}\| \to 0$. Since $\chi_{\{-N,\ldots,N\}} u \to u$ in ℓ^2 as $N \to \infty$ and since $H_W - t$ is a continuous operator, we may in fact also assume that the $u^{(n)}$ have finite supports.

Since $L_n \to \infty$, there are $n_j \to \infty$ so that $u^{(j)}$ is supported by $\{-L_{n_j}, \ldots, L_{n_j}\}$. To keep the notation simple, we will just assume that $n_j = j$ works.

Then the shifted sequence $v_j^{(n)} = u_{j-c_n}^{(n)}$ is a Weyl sequence: the $v^{(n)}$ have disjoint supports, so form an ONS, and hence $v^{(n)} \xrightarrow{w} 0$. Moreover, $\|(H_V^+ - t)v^{(n)}\| = \|(H_W - t)u^{(n)}\| \to 0$. $\square$

These results give information on the spectrum as a set. We are also interested in finer properties of the spectrum, such as the ac, sc, pp decomposition. We start with rank one perturbations, and we will in fact again work in an abstract framework, for general Hilbert space operators. So let $T \in B(H)$ be self-adjoint, and assume that T has simple spectrum. Fix a cyclic vector $x \in H$, $\|x\| = 1$. Recall that this means that $\{f(T)x : f \in C(\sigma(T))\}$ is dense in H. We want to consider the family of rank one perturbations

$$T_g = T + g\langle x, \cdot \rangle x \qquad (g \in \mathbb{R}).$$

The following observations confirm that this is a good choice of setup.

Exercise 15.4. Let $T \in B(H)$ be normal, and let $M \subset H$ be a closed subspace. Show that M is reducing if and only if M is invariant under both T and T^*.

Now suppose that we are given an arbitrary self-adjoint operator $T \in B(H)$ and an arbitrary vector $x \in H$, $\|x\| = 1$. Form the subspace $H_1 = \overline{\{f(T)x : f \in C(\sigma(T))\}}$. Then H_1 is clearly invariant under T, thus reducing by the Exercise. Thus we can decompose $T = T_1 \oplus T_2$, where $T_2 : H_1^\perp \to H_1^\perp$. Then

$$T + g\langle x, \cdot \rangle x = (T_1 + g\langle x, \cdot \rangle x) \oplus T_2.$$

Since it is also clear that x is cyclic for T_1, we have reduced the situation of a general rank one perturbation to the one outlined above.

We also discover such a scenario in the theory of Jacobi matrices: If $T = J$, a Jacobi matrix on $H = \ell^2(\mathbb{N})$, then $x = \delta_1$ is a cyclic vector. Note that the perturbed operator $J_g = J + g\langle \delta_1, \cdot\rangle\delta_1$ is again a Jacobi matrix. In fact, we obtain it from J by simply replacing $b_1 \to b_1 + g$.

Proposition 15.8. *For every $g \in \mathbb{R}$, x is a cyclic vector for T_g.*

Proof. An inductive argument shows that $T_g^n x = T^n x + y$, where y is a linear combination of $x, Tx, \ldots, T^{n-1}x$. So $L(x, Tx, \ldots, T^n x) = L(x, T_g x, \ldots, T_g^n x)$, or, put differently, $\{p(T)x\} = \{p(T_g)x\}$, where p varies over all polynomials. However, every continuous function on the compact set $\sigma(T) \subset \mathbb{R}$ can be uniformly approximated by polynomials, so $\{p(T)x\}$ is already dense in H. $\square$

Since x is cyclic, we know from Theorem 10.7 and its proof that T_g is unitarily equivalent to multiplication by t on $L^2(\mathbb{R}, \mu_g)$, where $d\mu_g(t) = d\|E_g(t)x\|^2$, and here E_g of course denotes the spectral resolution of T_g. By the functional calculus,

$$F_g(z) \equiv \langle x, (T_g - z)^{-1}x\rangle = \int_{\mathbb{R}} \frac{d\mu_g(t)}{t - z} \qquad (z \notin \mathbb{R}).$$

These functions F_g satisfy the following identity, which will be crucial for everything that follows.

Theorem 15.9.

$$F_g(z) = \frac{F(z)}{1 + gF(z)} \tag{15.1}$$

Here, $F(z) = F_0(z) = \langle x, (T - z)^{-1}x\rangle$.

Proof. Write $P = \langle x, \cdot\rangle x$ and notice that (for $z \notin \mathbb{R}$)

$$(T_g - z)^{-1} - (T - z)^{-1} = -g(T_g - z)^{-1}P(T - z)^{-1},$$

so

$$\begin{aligned} F_g(z) - F(z) &= -g\langle x, (T_g - z)^{-1}P(T - z)^{-1}x\rangle \\ &= -g\langle x, (T_g - z)^{-1}x\rangle\langle x, (T - z)^{-1}x\rangle = -gF_g(z)F(z), \end{aligned}$$

and we obtain (15.1) by rearranging. $\square$

We can now use (15.1) to show that the ac part of a self-adjoint operator is invariant under rank one perturbations. We need some preliminary observations. Let ρ be an absolutely continuous (positive) Borel measure on $\mathbb{R}$. For simplicity, we also assume that ρ is finite. Then a Borel set $M \subset \mathbb{R}$ is called an *essential support* of ρ if $\rho(M^c) = 0$

and if $N \subset M$, $\rho(N) = 0$, then $|N| = 0$, where $|\cdot|$ denotes Lebesgue measure. By the Radon-Nikodym Theorem, we can write $d\rho(x) = f(x)\,dx$, with $f \in L^1(\mathbb{R})$, $f \geq 0$, and now $M = \{x \in \mathbb{R} : f(x) > 0\}$ provides an essential support. Essential supports are unique up to null sets: If M, M' are essential supports, then $|M\Delta M'| = 0$, where $M\Delta M' = (M \setminus M') \cup (M' \setminus M)$. Moreover, essential supports determine the measure class, in the following sense: Let M_ρ, M_ν be essential supports of the (absolutely continuous) measures ρ, ν. Then ρ and ν are equivalent (have the same null sets) if and only if $|M_\rho \Delta M_\nu| = 0$, which happens if and only if ρ, ν have a common essential support M.

Recall from Exercise 10.17 that two simple self-adjoint operators S, T are unitarily equivalent if and only if they have equivalent spectral measures μ, ν. So we can now say that $S_{ac} \cong T_{ac}$ if and only if μ_{ac}, ν_{ac} admit a common essential support. With these preparations out of the way, it will now be an easy matter to establish the following fact:

Theorem 15.10. *T_g and T have unitarily equivalent absolutely continuous parts.*

This of course implies that $\sigma_{ac}(T_g) = \sigma_{ac}(T)$, but the actual statement is stronger than this because, in general, the ac spectra can be equal without the ac parts of the operators being unitarily equivalent.

Exercise 15.5. Explain this in more detail.
Suggestion: Construct two ac measures μ, ν, so that $M_t^{(\mu)}$ (in $L^2(\mu)$) and $M_t^{(\nu)}$ (in $L^2(\nu)$) have the same spectra, but are not unitarily equivalent. Equivalently, you need to construct two ac measures that have the same topological support but not the same null sets.

Proof. We work with the measures $d\mu_g(t) = d\|E_g(t)x\|^2$ that were introduced above. By Theorem 13.10(a), (c), $F_g(t) \equiv \lim_{y\to 0+} F_g(t+iy)$ exists for almost every $t \in \mathbb{R}$, and $d(\mu_g)_{ac}(t) = (1/\pi)\mathrm{Im}\, F_g(t)\,dt$. As discussed above, $M_g = \{t \in \mathbb{R} : \mathrm{Im}\, F_g(t) > 0\}$ is an essential support of this measure. Fix $g \in \mathbb{R}$ and assume that $t \in M = M_0$. By throwing away a null set $N \subset \mathbb{R}$, we may also assume that $F(t) = \lim F(t+iy)$ and $F_g(t)$ exist; since $t \in M$, we have that $\mathrm{Im}\, F(t) > 0$. From (15.1), we see that

$$\mathrm{Im}\, F_g(z) = \frac{\mathrm{Im}\, F(z)}{|1 + gF(z)|^2}.$$

Take $z = t + iy$ and let $y \to 0+$. It follows that $\mathrm{Im}\, F_g(t) > 0$, too. In terms of the supports, this calculation has shown that we can take $M_g \supset M$. By symmetry, we also obtain that $M_g \subset M$. $\square$

This result can be improved. First of all, any self-adjoint finite rank perturbation is of the form $V = \sum_{n=1}^N v_n \langle x_n, \cdot\rangle x_n$ and thus may be interpreted as N successive rank one perturbations. So the ac part of a self-adjoint operator is invariant, up to unitary equivalence, under (self-adjoint) finite rank perturbations. A stronger result holds, but this is not so easy to prove, so I'll just report on this:

Theorem 15.11 (Kato-Rosenblum)**.** *Suppose that $T \in B(H)$ is self-adjoint and V is self-adjoint and $V \in B_1(H)$. Then the ac parts of T and $T+V$ are unitarily equivalent.*

Exercise 15.6. Prove that the ac spectrum also obeys a decomposition law: If J is a Jacobi matrix on $\ell^2(\mathbb{Z})$, then $\sigma_{ac}(J) = \sigma_{ac}(J_+) \cup \sigma_{ac}(J_-)$ (the notation is as in Theorem 15.6).

The trace class condition in Theorem 15.11 is sharp. This is demonstrated by the following rather spectacular result (which we don't want to prove here).

Theorem 15.12 (Weyl-von Neumann)**.** *Let $T \in B(H)$ be a self-adjoint operator on a separable Hilbert space H. Then, for every $p > 1$ and $\epsilon > 0$, there exists a self-adjoint $K \in B_p(H)$ with $\|K\|_p < \epsilon$ so that $\sigma_{ac}(T+K) = \sigma_{sc}(T+K) = \emptyset$.*

So $T + K$ has pure point spectrum. Since the essential spectrum is preserved by the compact perturbation K, the closure of the eigenvalues of $T + K$ has to contain $\sigma_{ess}(T)$, so we will often get dense point spectrum here.

We have seen that the ac spectrum has reasonably good stability properties under small perturbations. What about the sc, pp parts? The following examples make short work of any hopes one might have. As a preparation, we first prove a criterion that will allow us to conveniently detect point spectrum.

Proposition 15.13. *Let*

$$G(x) = \int_{\mathbb{R}} \frac{d\mu(t)}{(x-t)^2} \in (0, \infty].$$

Then, for all $g \neq 0$, the following statements are equivalent:
(a) $\mu_g(\{x\}) > 0$;
(b) $G(x) < \infty$, $F(x) = -1/g$.

Here, $F(x) = -1/g$ could be interpreted as an abbreviation for the statement $F(x) = \lim_{y\to 0+} F(x+iy)$ exists and equals $-1/g$, but actually existence of this limit is automatic if $G(x) < \infty$.

Exercise 15.7. Prove this remark. More precisely, prove the following: If $G(x) < \infty$, then $F(x) = \lim_{y\to 0+} F(x + iy)$ exists and $F(x) \in \mathbb{R}$.

Proof. Recall that $\mu_g(\{x\}) = \lim -iyF_g(x+iy)$ (Theorem 13.10(e)). So, if $\mu_g(\{x\}) > 0$, then

$$F(x+iy) = \frac{F_g(x+iy)}{1-gF_g(x+iy)} = \frac{yF_g(x+iy)}{y-gyF_g(x+iy)} \to -\frac{1}{g}.$$

Moreover,

$$\frac{\operatorname{Im} F(x+iy)}{y} = \frac{y\operatorname{Im} F(x+iy)}{|y-gyF_g(x+iy)|^2}$$

also approaches a finite, positive limit as $y \to 0+$. On the other hand,

$$\frac{\operatorname{Im} F(x+iy)}{y} = \int_{\mathbb{R}} \frac{d\mu(t)}{y^2+(x-t)^2},$$

and this converges to $G(x)$ by the Monotone Convergence Theorem, so $G(x) < \infty$.

Conversely, if $G(x) < \infty$, then the same calculation shows that $\operatorname{Im} F(x+iy)/y \to G(x)$. Moreover, $1/|t-x| \in L^1(\mu)$, so Dominated Convergence shows that

$$F(x) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-x}$$

(compare Exercise 15.7). Hence

$$\frac{F(x+iy)-F(x)}{y} = i\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x-iy)(t-x)} \to iG(x),$$

by Dominated Convergence again. In other words, if also $F(x) = -1/g$, then $(1+gF(x+iy))/y \to igG(x)$. It now follows that

$$y\operatorname{Im} F_g(x+iy) = \frac{y^{-1}\operatorname{Im} F(x+iy)}{y^{-2}|1+gF(x+iy)|^2} \to \frac{1}{g^2G(x)} > 0,$$

so $\mu_g(\{x\}) > 0$, as claimed. $\square$

In the following examples, we will just give the measure μ. This will determine the measures μ_g completely, via F, F_g and (15.1). Moreover, we can just let $H = L^2(\mathbb{R}, d\mu)$, $T = M_t$, $x \equiv 1$ to confirm that there indeed is a self-adjoint operator and a cyclic vector for which this measure μ is the corresponding spectral measure. Alternatively, we could let $T = J$ be the Jacobi matrix with spectral measure μ (use Theorem 13.9!) and $x = \delta_1$.

Example 15.1. Let $d\mu(x) = (1/2)\chi_{[0,1]}(x)\,dx + \sum_{n\ge 1} 2^{-n-1}\delta_{x_n}$, where x_n is a countable dense subset of $[0,1]$. Then $\sigma_{ac}(T) = \sigma_{pp}(T) = [0,1]$. However, for all $0 \le x \le 1$, we have that

$$G(x) \ge \frac{1}{2}\int_0^1 \frac{dt}{(x-t)^2} = \infty,$$

so $\sigma_{pp}(T_g) \cap [0,1] = \emptyset$ for all $g \neq 0$ by Proposition 15.13.

Example 15.2. Let $\rho_n = 2^{-n} \sum_{j=1}^{2^n} \delta_{j2^{-n}}$ and $\mu = \sum_{n\geq 1} 2^{-n}\rho_n$. Then $\sigma_{pp}(T) = [0,1]$, $\sigma_{ac}(T) = \sigma_{sc}(T) = \emptyset$. If $x \in [0,1]$, then there is a j so that $|x - j2^{-n}| \leq 2^{-n}$, so

$$\int_{\mathbb{R}} \frac{d\rho_n(t)}{(x-t)^2} \geq 2^{-n}2^{2n} = 2^n,$$

and $G(x) \geq \sum 2^{-n}2^n = \infty$. Proposition 15.13 shows that $\sigma_{pp}(T_g) \cap [0,1] = \emptyset$ for all $g \neq 0$. Since both the essential and the ac spectrum are preserved, it follows that $\sigma_{sc}(T_g) = [0,1]$.

Exercise 15.8. Show that $\sigma_p(T_{g_1}) \cap \sigma_p(T_{g_2}) = \emptyset$ if $g_1 \neq g_2$ (we are working with σ_p here, the set of eigenvalues, not its closure σ_{pp}).

Exercise 15.9. Let J be a Jacobi matrix on $\ell^2(\mathbb{N})$, and let $d\mu(t) = d\|E(t)\delta_1\|^2$, as usual. Consider the family of rank one perturbations J_g (corresponding to the coefficient change $b_1 \to b_1 + g$).
(a) Show that $x \in \mathbb{R}$ is an eigenvalue of J_g for some $g \in \mathbb{R}$ if and only if $(\tau - x)u = 0$ has an ℓ^2 solution u with $u_1 \neq 0$.
(b) Show that $G(x) < \infty$ if and only if $(\tau - x)u = 0$ has an ℓ^2 solution u with $u_0, u_1 \neq 0$.

Exercise 15.10. Let $T \in B(H)$ be a self-adjoint operator. Show that T is compact if and only if $\sigma_{ess}(T) \subset \{0\}$.