14. Compact operators

We now return, for the time being, to the general theory of linear operators. Let \( X, Y \) be Banach spaces, and let \( T : X \to Y \) be a linear operator (defined everywhere).

**Definition 14.1.** \( T \) is called **compact** if \( \overline{T(B)} \) is a compact set; here, \( B = B_1(0) = \{ x \in X : \| x \| < 1 \} \). We denote the set of compact operators by \( B_\infty(X,Y) \); if \( X = Y \), we write \( B_\infty(X) \) instead of \( B_\infty(X,X) \).

**Exercise 14.1.** Let \( T \in B_\infty(X,Y) \). Show that \( T(B) \) is compact for any bounded set \( B \subset X \).

Since compact sets are bounded, it follows that compact operators are always bounded: \( B_\infty(X,Y) \subset B(X,Y) \). Here’s a convenient rephrasing of the definition.

**Proposition 14.2.** \( T : X \to Y \) is compact if and only if every sequence \( x_n \in B \) has a subsequence \( x_{n_j} \) for which \( Tx_{n_j} \) converges.

**Exercise 14.2.** Prove the Proposition.

**Theorem 14.3.** Suppose that \( S, T \in B_\infty(X), A \in B(X), \) and \( c \in \mathbb{C} \). Then \( S + T, cT, AT, TA \in B_\infty(X) \).

Put differently, this says that \( B_\infty(X) \subset B(X) \) is a two-sided ideal in the Banach algebra \( B(X) \) ("two-sided" refers to the fact that we may multiply by \( A \in B(X) \) from either side).

**Proof.** We verify the criterion from Proposition 14.2. Given a sequence \( x_n \in B \), pick a subsequence \( x'_n \) so that \( Sx'_n \) converges and then a sub-subsequence \( x''_n \) so that \( Tx''_n \) converges, too. Then \( (S+T)x''_n, cTx''_n, \) and \( ATx''_n \) converge. Furthermore, since \( A \) is bounded, \( Ax_n \) is just another bounded sequence, so \( T(Ax_n) \) can also be made convergent by passing to a subsequence. \( \square \)

**Theorem 14.4.** \( B_\infty(X) \) is a closed subset of \( B(X) \).

So \( I = B_\infty(X) \) is a closed two-sided ideal of \( B(X) \). If \( X = H \) is a separable Hilbert space, then it can be shown that \( B_\infty(H) \) is the only closed two-sided ideal \( \neq \{0\}, H \).

**Proof.** Suppose that \( T_n \in B_\infty(X), T \in B(X), \|T_n - T\| \to 0 \), and let \( x_n \) be a sequence from \( B \). We must show that \( Tx_n \) has a convergent subsequence. For fixed \( m \), we can of course make \( T_m x_n \) convergent as \( n \to \infty \) by passing to a suitable subsequence, and we can do better than this: a diagonal process lets us find a subsequence \( x'_n \) with the property that \( \lim_{n \to \infty} T_m x'_n \) exists for all \( m \).
Now if \( \epsilon > 0 \) is given, fix an \( n \in \mathbb{N} \) with \( \| T_n - T \| < \epsilon \). Then take \( N \in \mathbb{N} \) so large that (for this \( n \)) \( \| T_n(x'_j - x'_k) \| < \epsilon \) for all \( j, k \geq N \). For these \( j, k \), we then also have that

\[
\| T(x'_j - x'_k) \| \leq \| T_n(x'_j - x'_k) \| + \| T_n - T \| \cdot \| x'_j - x'_k \| < \epsilon + 2\| T_n - T \| < 3\epsilon,
\]

so \( Tx'_n \) is a Cauchy sequence and thus convergent. \( \square \)

We say that \( T \in B(X, Y) \) is a finite rank operator if \( \dim R(T) < \infty \). In this case, if \( x_n \in B \), then \( Tx_n \) is a bounded sequence from the finite-dimensional space \( R(T) \cong \mathbb{C}^N \), so we can extract a convergent subsequence by the classical Bolzano-Weierstraß Theorem. Recall also that all norms on a finite-dimensional space are equivalent, so it suffices to identify \( R(T) \) with \( \mathbb{C}^N \) as a vector space and then automatically the induced topology must be the usual topology on \( \mathbb{C}^N \).

So every finite rank operator is compact. In particular, \( B(\mathbb{C}^n) = B_\infty(\mathbb{C}^n) \). Further examples of compact operators are provided by the following Exercise.

**Exercise 14.3.** Suppose that \( t_n \to 0 \), and let \( T : \ell^p \to \ell^p \) (1 \( \leq p \leq \infty \)) be the operator of multiplication by \( t_n \). More precisely, \((Tx)_n = t_n x_n \). Show that \( T \) is compact.

**Suggestion:** Consider the finite rank truncations \( T_N \) corresponding to the truncated sequence \( t_n^{(N)} \) and use Theorem 14.4; here, \( t_n^{(N)} = t_n \) if \( n \leq N \) and \( t_n^{(N)} = 0 \) if \( n > N \).

We now focus on compact operators on a Hilbert space \( H \).

**Theorem 14.5.** Let \( T \in B(H) \). Then

\[
T \in B_\infty(H) \iff T^* \in B_\infty(H) \iff T^* T \in B_\infty(H).
\]

**Proof.** Theorem 14.3 shows that \( T^* T \in B_\infty(H) \) if \( T^* \in B_\infty(H) \) (or \( T \in B_\infty(H) \)).

Next, assume that \( T^* T \in B_\infty(H) \), and let \( x_n \in B \). Then \( T^* T x_n \) converges on a suitable subsequence, which, for convenience, we will again denote by \( x_n \). The following calculation shows that \( T x_n \) converges on the same subsequence, so \( T \in B_\infty(H) \).

\[
\| T(x_m - x_n) \|^2 = \langle T(x_m - x_n), T(x_m - x_n) \rangle = \langle T^* T(x_m - x_n), x_m - x_n \rangle \leq \| T^* T(x_m - x_n) \| \| x_m - x_n \| \leq 2\| T^* T(x_m - x_n) \|
\]

Finally, if \( T \) is compact, then \( TT^* = T^{**} T^* \in B_\infty(H) \) by Theorem 14.3 again, so the argument from the preceding paragraph now shows that \( T^* \in B_\infty(H) \), too. \( \square \)
Exercise 14.4. Let \( P \in B(H) \) be the projection onto the subspace \( M \subset H \). Show that \( P \) is compact if and only if \( \dim M < \infty \).

Compactness of operators on a Hilbert space admits an especially neat sequence characterization.

**Theorem 14.6.** Let \( T : H \to H \) be a linear operator (with \( D(T) = H \)).

(a) The following statements are equivalent:
   (i) \( T \in B(H) \);
   (ii) \( x_n \to 0 \implies Tx_n \to 0 \);
   (iii) \( x_n \rightharpoonup 0 \implies Tx_n \rightharpoonup 0 \);
   (iv) \( x_n \to 0 \implies Tx_n \rightharpoonup 0 \)

(b) The following statements are equivalent:
   (i) \( T \in B_\infty(H) \);
   (ii) \( x_n \rightharpoonup 0 \implies Tx_n \to 0 \)

Here, we of course need to remember that \( x_n \rightharpoonup x \) if and only if \( \langle y, x_n \rangle \to \langle y, x \rangle \) for all \( y \in H \).

Exercise 14.5. Let \( x_n \in H \) and suppose that \( \lim_{n \to \infty} \langle y, x_n \rangle \) exists for every \( y \in H \). Show that then \( x_n \) is bounded, that is, there exists \( C > 0 \) so that \( \|x_n\| \leq C \) for all \( n \in \mathbb{N} \).

*Hint:* Apply the uniform boundedness principle to the maps \( F_n(y) = \langle x_n, y \rangle \).

Note that every weakly convergent sequence \( x_n \) satisfies the assumption from this Exercise; conversely, it can be shown that such a sequence \( x_n \) is weakly convergent, so we could have assumed this instead. The version given here will prove useful in a moment.

In the proof of Theorem 14.6, we will need the following Lemma, which is of considerable independent interest.

**Lemma 14.7.** Every bounded sequence \( x_n \in H \) has a weakly convergent subsequence.

*Proof.* For every fixed \( m \), the sequence \( (\langle x_m, x_n \rangle)_n \) is a bounded sequence of complex numbers, so it has a convergent subsequence by the Bolzano-Weierstraß Theorem. Again, a diagonal process lets us in fact find a subsequence \( x'_n \) for which \( \langle x_m, x'_n \rangle \) converges, as \( n \to \infty \), for all \( m \). The (anti-)linearity of the scalar product now implies that \( \lim \langle y, x'_n \rangle \) exists for all \( y \in L(x_m) \).

Exercise 14.6. Show that this limit exists for all \( y \in \overline{L(x_m)} \).

*Suggestion:* Show that the scalar products form a Cauchy sequence.
Finally, if \( w \in H \) is arbitrary, write \( w = y + z \) with \( y \in M = \overline{\text{L}(x_m)} \) and \( z \in M^\perp \). Then \( \langle w, x'_n \rangle = \langle y, x'_n \rangle \), so this sequence converges, too.

To show that \( x'_n \) is weakly convergent, we still need to produce an \( x \in H \) so that \( \lim \langle w, x'_n \rangle = \langle w, x \rangle \) for all \( w \in H \). To do this, notice first of all that the sequence \( x'_n \) is bounded, by Exercise 14.5. Therefore, the linear functional \( F(w) = \lim \langle x'_n, w \rangle \) is bounded: \( |F(w)| \leq \limsup \|x'_n\| \|w\| \leq C \|w\| \). The Riesz Representation Theorem now shows that \( F(w) = \langle x, w \rangle \) for some \( x \in H \), as desired. \( \Box \)

**Proof of Theorem 14.6.** (a) (i) \( \implies \) (ii): This is obvious, because (ii) is just the sequence version of continuity at \( x = 0 \), and so (i) and (ii) are in fact equivalent.

(ii) \( \implies \) (iii): As just observed, \( T \in B(H) \). If \( x_n \overset{w}{\to} 0 \), then also

\[
\langle y, Tx_n \rangle = \langle T^* y, x_n \rangle \to 0
\]

for all \( y \in H \), so \( Tx_n \overset{w}{\to} 0 \).

(iii) \( \implies \) (iv) is trivial.

(iv) \( \implies \) (i): Suppose that \( T \notin B(H) \). Then we can find \( x_n \in H \), \( \|x_n\| = 1 \), with \( \|Tx_n\| \geq n^2 \). Let \( y_n = (1/n)x_n \). Then \( y_n \to 0 \), but \( \|Ty_n\| \geq n \), so, by Exercise 14.5, the sequence \( Ty_n \) cannot be weakly convergent.

(b) (i) \( \implies \) (ii): Let \( x_n \in H \), \( x_n \overset{w}{\to} 0 \). Then \( x_n \) is bounded (Exercise 14.5 again), so there exists a subsequence for which \( Tx'_n \) converges, say \( Tx'_n \to y \). In particular, \( Tx'_n \overset{w}{\to} y \), and now part (a), condition (iii) shows that we must have \( y = 0 \) here. This whole argument has in fact shown that every subsequence \( x'_n \) of \( x_n \) has a sub-subsequence \( x''_n \) so that \( Tx''_n \to 0 \). It follows that \( Tx_n \to 0 \), without the need of passing to a subsequence.

(ii) \( \implies \) (i): Let \( x_n \in B \). By Lemma 14.7, we can extract a weakly convergent subsequence, which we denote by \( x_n \) also. So \( x_n \overset{w}{\to} x \), and thus \( x_n - x \overset{w}{\to} 0 \). By hypothesis, \( T(x_n - x) \to 0 \), so indeed \( Tx_n \) converges (to \( Tx \)). \( \Box \)

We now discuss the spectral theory of compact operators. We first deal with compact normal operators. The following two results give a complete spectral theoretic characterization of these.

**Theorem 14.8.** Let \( T \in B(H) \) be a compact, normal operator. Then \( \sigma(T) \) is countable. Write \( \sigma(T) \setminus \{0\} = \{z_n\} \). Then each \( z_n \) is an eigenvalue of \( T \) of finite multiplicity: \( 1 \leq \dim N(T - z_n) < \infty \). Moreover, \( z_n \to 0 \) if \( \{z_n\} \) is infinite.
If $P_n$ denotes the projection onto the eigenspace $N(T - z_n)$, then

\[(14.1)\quad T = \sum z_n P_n.\]

This series converges in $B(H)$, for an arbitrary arrangement of the $z_n$. Finally, if $\dim H = \infty$, then $0 \in \sigma(T)$.

**Proof.** Denote the open disk about 0 of radius $r$ by $D_r = \{ z \in \mathbb{C} : |z| < r \}$, and let $P = E(D_r^c)$, where $E$ is the spectral resolution of $T$. Let $M = R(P)$, which is a reducing subspace for $T$ by Exercise 10.22. I claim that $\dim M < \infty$. Indeed, if this were wrong, we could find a sequence $x_n \in M$, $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ (pick any ONS in $M$). Theorem 14.6(b) then shows that $Tx_n \to 0$. This, however, is impossible because the functional calculus shows that $\|Tx_n\|^2 = \int |z|^2 d\|E(z)x_n\|^2 \geq r^2 > 0$.

Now since $M$ is reducing, we can decompose $T = T_M \oplus T_{M^\perp}$, and $M^\perp = R(E(D_r))$, so $\|T_M\| \leq r$, and thus $T_M - z$ is definitely invertible in $B(M^\perp)$ if $|z| > r$. So such a $z$ will be in $\rho(T)$, unless $z \in \sigma(T_M)$, but $T_M$ is an operator on the finite-dimensional space $M$, so its spectrum consists of eigenvalues only, and there are only finitely many of these. Conversely, it is clear that every eigenvalue of $T_M$ is an eigenvalue of $T$ also, so we have shown the following: $\sigma(T) \cap D_r$ is finite for every $r > 0$ and contains only eigenvalues of $T$. Moreover, these are of finite multiplicity because $N(T - z) = E(\{z\}) \subset E(D_r^c) = M$.

It now follows that $\sigma(T)$ is countable, and we also obtain the statements about the sequence $z_n$. If $\dim H = \infty$, then either $E(\{0\}) \neq 0$ or the sequence $z_n$ is infinite and thus converges to 0. In both cases, $0 \in \sigma(T)$.

It remains to establish (14.1). Notice that $P_n = E(\{z_n\})$; in particular, the $P_n$ have mutually orthogonal ranges. The Spectral Theorem shows that

\[(14.2)\quad \langle x, Ty \rangle = \int \mathbb{C} z d\mu_{x,y}(z) = \sum z_n \langle x, P_n y \rangle.\]

In the second step, we use the fact that $E$ is supported by $\{z_n\} \cup \{0\}$, so $\mu_{x,y}$ is a density times counting measure on this set and thus the integral is a sum. Next, we verify that (14.1) converges in $B(H)$. More precisely, we will prove that the partial sums form a Cauchy sequence.
Let \( x \in H \), and consider

\[
\left\| \sum_{n=N+1}^{N'} z_n P_n x \right\|^2 = \sum_{n=N+1}^{N'} |z_n|^2 \|P_n x\|^2 \leq \left( \sup_{n>N} |z_n|^2 \right) \cdot \sum_{n=N+1}^{N'} \|P_n x\|^2
\]

\[
\leq \left( \sup_{n>N} |z_n|^2 \right) \cdot \|x\|^2.
\]

This implies that

\[
\left\| \sum_{n=1}^{N'} z_n P_n - \sum_{n=1}^{N} z_n P_n \right\| \leq \sup_{n>N} |z_n|,
\]

and this supremum goes to zero as \( N \to \infty \), as desired.

So the right-hand side of (14.1) has a limit, and now (14.2) shows that this limit must be \( T \).

So normal compact operators have representations of the type (14.1). It is also true that, conversely, if we are given data \( z_n \) and \( P_n \) with the properties stated in the Theorem, then we can use (14.1) to define a normal compact operator \( T \). In other words, (14.1) for sequences \( z_n \to 0 \) and mutually orthogonal finite-dimensional projections \( P_n \) lists exactly all normal compact operators.

To formulate this converse, we slightly change the notation. We let \( \langle x, \cdot \rangle x \) denote the operator that maps \( y \mapsto \langle x, y \rangle x \).

**Exercise 14.7.** Show that \( \langle x, \cdot \rangle x = \|x\|^2 P_{L(x)} \). Also, show that if \( \{x_1, \ldots, x_N\} \) is an ONB of the (finite-dimensional) subspace \( M \), then

\[
P_M = \sum_{n=1}^{N} \langle x_n , \cdot \rangle x_n.
\]

**Theorem 14.9.** Let \( \{x_n\} \) be an ONS, and let \( z_n \in \mathbb{C}, z_n \neq 0, z_n \to 0 \) (if the sequence is infinite). Then the series

\[
T = \sum z_n \langle x_n , \cdot \rangle x_n
\]

converges in \( B(H) \) (if infinite) to a compact normal operator \( T \). We have that \( \sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} = \{z_n\} \).

Note that Exercise 14.7 guarantees that the series from (14.1) are of this form; if \( \dim R(P_n) > 1 \), then we need to pick an ONB of this space and repeat the corresponding eigenvalue \( z_n \) that number of times.

**Proof.** By Exercise 14.7, the operators \( \langle x_n , \cdot \rangle x_n \) are projections onto the mutually orthogonal subspaces \( L(x_n) \), so convergence of the series in \( B(H) \) follows as in the previous proof. For each fixed \( N \), the operator...
\[ \sum_{n=1}^{N} \omega_n \langle x_n, \cdot \rangle x_n \text{ is of finite rank, thus compact, and hence } T \text{ is compact by Theorem 14.4.} \]

To prove that \( T \) is normal, we temporarily change our notation again and write \( \langle x_n, \cdot \rangle x_n = P_n \). We compute

\[ TT^* = \lim_{N \to \infty} \sum_{n=1}^{N} \omega_m P_m \sum_{n=1}^{N} \overline{\omega_n} P_n = \lim_{N \to \infty} \sum_{n=1}^{N} |\omega_n|^2 P_n \]

\[ = \lim_{N \to \infty} \sum_{n=1}^{N} \overline{\omega_n} P_n \sum_{m=1}^{N} \omega_m P_m = T^*T, \]

so \( T \) is normal.

It is also clear that \( Tx_n = \omega_n x_n \), and since \( T \) is compact, any other non-zero point from the spectrum would have to be an eigenvalue, too, so the following Exercise finishes the proof.

**Exercise 14.8.** Show that if \( z \not\in \{\omega_n\} \cup \{0\} \), then \( Tx = zx \) has no solution \( x \neq 0 \).

\[ \square \]

We now move on to arbitrary compact operators \( T \in B(H) \), not necessarily normal. Actually, we are going to start with some introductory observations that apply to arbitrary bounded operators \( T \in B(H) \). We will consider \( T^*T \), and this is a positive operator by Theorem 9.15.

**Exercise 14.9.** Give an easier proof of this statement \( (T^*T \geq 0 \text{ if } T \in B(H)) \) that is based on Theorem 10.13.

By Theorem 10.14, \( T^*T \) has a unique positive square root, which we will denote by \( |T| := (T^*T)^{1/2} \).

**Exercise 14.10.** Show that if \( T \) is normal, then this definition of \( |T| \) coincides with the one obtained from the functional calculus. In other words, show that

\[ |T| = \int_{\mathbb{C}} f(z) \, dE(z), \]

where \( E \) is the spectral resolution of \( T \) and \( f(z) = |z| \).

This operator \( |T| \) has the important property that

\[ \| T |x \| = \| Tx \| \]

for all \( x \in H \). We see this from the calculation

\[ \| T |x \| ^2 = \langle T |x, T |x \rangle = \langle x, |T|^2 x \rangle = \langle x, T^* Tx \rangle = \langle Tx, Tx \rangle = \| Tx \|^2. \]
Exercise 14.11. Compute $|T|$ for
\[
T = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1/\sqrt{2} & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}.
\]

Theorem 14.10. Let $T$ be a compact operator. Then $|T|$ is also compact. Moreover, there exists a unique unitary map $V : R(|T|) \to R(T)$ so that $T = V|T|$.

This representation $T = V|T|$ is called the polar decomposition of $T$. This terminology emphasizes the analogy to the polar representation of complex numbers $z = e^{i\varphi}|z|$.

Proof. $T^*T$ is compact by Theorem 14.3 (or Theorem 14.5). So Theorem 14.8 gives a representation of the type $T^*T = \sum t_n P_n$. Since $T^*T \geq 0$, we must have that $t_n > 0$; if the sequence is infinite, then $t_n \to 0$. It now follows that $|T| = \sum t_n^{1/2} P_n$ (positive square roots) because this operator is positive by Theorem 14.9 (its spectrum consists of the $t_n^{1/2}$ and possibly 0) and its square equals $T^*T$, and positive square roots are unique. Theorem 14.9 then also shows that $|T| \in B_\infty(H)$.

To construct $V$, define $V_0 : R(|T|) \to R(T)$ by $V_0(|T|x) = Tx$. This is indeed well defined because if $|T|x = |T|y$, then $|T|(x-y) = 0$, so, by (14.3), $T(x-y) = 0$, that is, $Tx = Ty$. Moreover, (14.3) also shows that $V_0$ is isometric. In particular, $V_0$ is continuous, and thus there exists an isometric extension to $\overline{R(|T|)}$. Since $R(V_0) = R(T)$ and isometries have closed ranges, it follows that $R(V) = \overline{R(T)}$. By the construction of $V_0$, we have the identity $V_0|T| = T$, so $V|T| = T$ (note that $|T|x \in R(|T|)$ for all $x$, so as far as this identity is concerned, it doesn’t matter if or how we extend $V_0$).

Finally, if also $W|T| = T$, then the restriction of $W$ to $R(|T|)$ must agree with $V_0$, and there is only one continuous extension to the closure, so $W = V$ and $V$ is unique.

To obtain series representations for arbitrary compact operators, we introduce additional data. Let $s_1(T) \geq s_2(T) \geq s_3(T) \geq \ldots > 0$ be the non-zero eigenvalues of $|T|$ (what we called $t_n^{1/2}$ in the previous proof), repeated according to their (finite) multiplicities. The $s_n(T)$ are called the singular values of $T$. If the sequence of singular values is infinite, then $s_n(T) \to 0$.

Theorem 14.11. Let $T \in B_\infty(H)$. Then $s_n(T) = s_n(T^*) = s_n(|T|) = s_n(|T^*|)$. Moreover, there exist ONS $\{x_n\}$ and $\{y_n\}$, consisting of eigenvectors of $|T|$ and $|T^*|$, respectively (more precisely, $|T|x_n = s_n x_n$,
\[ |T^*|y_n = s_n y_n, \text{ so that} \]
\[ |T| = \sum s_n \langle x_n, \cdot \rangle x_n, \quad |T^*| = \sum s_n \langle y_n, \cdot \rangle y_n \]
\[ T = \sum s_n \langle x_n, \cdot \rangle y_n, \quad T^* = \sum s_n \langle y_n, \cdot \rangle x_n. \]

These sums converge in \( B(H) \) (if they are infinite).

**Proof.** We see as in the proof of Theorem 14.8 that these series converge in \( B(H) \) if \( \{x_n\}, \{y_n\} \) are (arbitrary) ONS. From this Theorem, we also know that \( |T| \) can indeed be written in this way, if we interpret \( s_n = s_n(T) \) and \( |T|x_n = s_n x_n \). Also, from the definition of the singular values, it is already clear that \( s_n(T) = s_n(|T|) \) and \( s_n(T^*) = s_n(|T^*|) \).

If we again let \( \{x_n\} \) be an ONS of eigenvectors of \( |T| \) (so \( |T|x_n = s_n x_n \)), then Theorem 14.10 shows that
\[ Tx = V |T| x = V \sum s_n \langle x_n, x \rangle x_n = \sum s_n \langle x_n, x \rangle y_n; \]
here, we have put \( y_n = V x_n \). Since \( x_n \) is an ONS from \( R(|T|) \) and \( V \) is unitary on this space, \( y_n \) is an ONS, too. Moreover, for arbitrary \( x, y \in H \), we have that
\[ \langle x, T^* y \rangle = \langle Tx, y \rangle = \sum s_n \langle x_n, x \rangle \langle y_n, y \rangle = \sum s_n \langle x_n, x \rangle \langle y_n, y \rangle \]
\[ = \langle x, \sum s_n \langle y_n, y \rangle x_n \rangle. \]
This establishes the formula for \( T^* \), except that we haven’t shown yet that the \( y_n \)’s are eigenvectors of \( |T^*| \). A similar calculation reveals that
\[ TT^* y = T \left( \sum s_n \langle y_n, y \rangle x_n \right) = \sum s_m s_n \langle y_n, y \rangle \langle x_m, x_n \rangle y_m \]
\[ = \sum s_n^2 \langle y_n, y \rangle y_n. \]
This says that \( |T^*| = \sum s_n \langle y_n, \cdot \rangle y_n \), and this formula clarifies everything: First of all, the \( s_n = s_n(T) \) are indeed the eigenvalues of \( |T^*| \), so \( s_n(T) = s_n(T^*) \). Moreover, we also see that the \( y_n \) are eigenvectors corresponding to these eigenvalues, and we obtain the asserted formula for \( |T^*| \). \( \square \)

**Corollary 14.12.** Let \( T \in B(H) \). Then \( T \) is compact if and only if there are finite rank operators \( T_n \in B(H) \) so that \( \| T_n - T \| \to 0 \).

**Proof.** Finite rank operators are compact, so one direction follows from Theorem 14.4. Conversely, if \( T \) is compact, then \( T = \sum s_n \langle x_n, \cdot \rangle y_n \), and the partial sums \( T_N = \sum_{n=1}^N s_n \langle x_n, \cdot \rangle y_n \) from a sequence of finite rank operators that converges to \( T \) in operator norm. \( \square \)
The singular values can be used to introduce subclasses of compact operators. More precisely, for \(1 \leq p < \infty\), let
\[
B_p(H) = \{ T \in B_\infty(H) : s_n(T) \in \ell^p \}.
\]
In fact, this is consistent with our notation \(B_\infty(H)\) for the compact operators and it also finally makes this choice of symbol more transparent. The spaces \(B_p\) are sometimes called von Neumann-Schatten classes or trace ideals. Of particular interest are \(B_2(H)\), the Hilbert-Schmidt operators, and \(B_1(H)\), the trace class operators. It can be shown that \(B_p(H)\) is in fact a Banach space with the norm \(\|T\|_p = \|s_n(T)\|_{\ell^p}\) (that this indeed defines a norm is not obvious, either). Much more could be said, but we will not pursue these topics here.

**Exercise 14.12.** Prove that if \(T\) is compact, then \(\|T\| = s_1(T)\). So \(\|T\|_\infty = \|T\| (= \|T\|_{B(H)})\) and \(\|T\| \leq \|T\|_p\) for all \(1 \leq p \leq \infty\).

**Exercise 14.13.** Consider the operator \(T \in B(\ell^2)\) that is given by
\[
(Tx)_n = \begin{cases} 0 & n = 1 \\ \frac{x_{n-1}}{n} & n \geq 2 \end{cases}.
\]
(a) Prove that \(T\) is compact.
(b) Prove that \(\sigma(T) = \{0\}, \sigma_p(T) = \emptyset\).

**Exercise 14.14.** Consider again the operator \(T\) from Exercise 14.13. Find \(T^*\) and \(|T|\) and prove that \(s_n(T) = \frac{1}{n+1}\) (so, in particular, \(T \in B_p\) for \(p > 1\), but \(T \notin B_1\)).

**Exercise 14.15.** Consider again the multiplication operator \((Tx)_n = t_n x_n\) on \(\ell^2\) from Exercise 14.3. Show that \(T \in B_1\) if and only if \(\sum |t_n| < \infty\).

**Exercise 14.16.** Let \(\mu\) be a finite Borel measure on \([0, 1]\), and let \(K : [0, 1] \times [0, 1] \to \mathbb{C}\) be a continuous function. Show that the operator \(T : L^2([0, 1], \mu) \to L^2([0, 1], \mu),\)
\[
(Tf)(x) = \int_{[0,1]} K(x, y) f(y) \, d\mu(y)
\]
is compact.