Solutions homework # 3: Complex Analysis 1

1. (a) If a_n is bounded, then this follows from Bolzano-Weierstraß. If a_n is unbounded, then we can extract a subsequence that converges to ∞ or $-\infty$.

   (b) I'll do this under the additional assumption that $A \in \mathbb{R}$. Essentially the same arguments also work if $A = \pm\infty$.

   Given any $k \geq 1$, we can find an accumulation point $a > A - 1/k$. It follows that there is an $a_{n_k} \in (a - 1/k, a + 1/k) \subseteq (A - 2/k, A + 2/k)$. So $a_{n_k} \to A$, and this says that A is an accumulation point, as claimed.

   (c) Let me first show that A, as defined in part (a), has these properties. Property (i) is clear from the fact that A is an accumulation point. If we had an infinite subsequence $a_{n_k} > B > A$, then a suitable sub-subsequence would converge to an (extended) limit $a \geq B > A$, but this is impossible because A is the largest accumulation point. So (ii) holds.

   Conversely, if A satisfies (i), (ii), then A is an accumulation point: for any $\epsilon > 0$, the interval $(A - \epsilon, A + \epsilon)$ contains infinitely many a_n, as we see by combining (i), (ii). By (ii), A is the largest accumulation point.

   (d) Denote the expression on the RHS by C. For any k, we have that $\sup_{n \geq k} a_n \geq A$, because A is an accumulation point. So $C \geq A$. On the other hand, for any $\epsilon > 0$, we will (eventually) have that $\sup_{n \geq k} a_n \leq A + \epsilon$, for all sufficiently large k. This follows from (c)(ii). We now see that $C \leq A$.

   (e) The sequence $(\sup_{n \geq k} a_n)_{k \geq 1}$ is decreasing, so the inf equals the lim.

   (f) If $a_n \to a$, then a is the only accumulation point of this sequence, so $\liminf a_n = \limsup a_n (= a)$. Conversely, if this holds, then a_n has to converge to this common value: if not, then we would find an infinite subsequence with $|a_{n_k} - a| \geq \epsilon > 0$, but this would lead to another accumulation point, and the smallest accumulation would not be the same as the largest.

2. *Proof.* Suppose that we had $P = \{z_n : n \geq 1\}$. Since z_1 is not isolated, we can find other points in P. Select such a point and call it a_1. Also, let $r_1 = |z_1 - a_1|/2 > 0$.

   Next, since $a_1 \in P$ is not isolated, the disk $D_1 = D(a_1, r_1)$ must contain other points of P. Pick one and call it a_2, and also take $r_2 > 0$ so small that $D(a_2, r_2) \subseteq D_1$ and $a_1 \notin \overline{D_2}$, and also $r_2 \leq r_1/2$.

   We continue in this style. In the next step, we find an $a_3 \neq a_2$, $a_3 \in D_2$, and from then on, we'll work in a disk D_3 about a_3 that is so small that it is contained in D_2 and, at the same time, $a_2 \notin \overline{D_3}$, and also $r_3 \leq r_2/2$.

   We obtain a sequence $a_n \in P$. This sequence is a Cauchy sequence because its tails $(a_k)_{k \geq n}$ lie in the disks D_n, and $r_n \to 0$. So $a_n \to a$, and here $a \in P$ since P is closed, but by construction, $a_n \not\to z_k$ for all k (at the first step, we move to a disk that is at a distance from z_1, then we avoid z_2 in the second step etc.). □