Homework # 3: Complex Analysis 1

1. Let $a_n \in \mathbb{R}$ and call $a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ an *accumulation point* if there exists a subsequence $a_{n_j} \to a$; here, convergence to $a = \pm\infty$ is defined in the expected way.

   (a) Show that a_n always has at least one accumulation point (in $\overline{\mathbb{R}}$!).

   (b) Show that
   $$A = \sup\{a \in \overline{\mathbb{R}} : a \text{ is an accumulation point of } a_n\}$$
   is an accumulation point itself. We now define $\limsup_{n\to\infty} a_n := A$.

   (c) Show that A is the unique (extended) number with the following properties: (i) If $B < A$, then $a_n > B$ for infinitely many $n \in \mathbb{N}$; (ii) if $B > A$, then $a_n > B$ for at most finitely many $n \in \mathbb{N}$.

   (d) Prove that
   $$A = \inf_{k\ge 1} \sup_{n\ge k} a_n.$$

   (e) Prove that also
   $$A = \lim_{k\to\infty} \sup_{n\ge k} a_n.$$

   (f) Prove that a_n converges (in the extended sense, in $\overline{\mathbb{R}}$) precisely if $\liminf a_n = \limsup a_n$, where we can define $\liminf a_n$ similarly, as the smallest accumulation point of $\{a_n\}$.

2. Let $P \subseteq \mathbb{C}$. Recall that $z \in P$ is called an *isolated point* if there is an $r > 0$ so that $D(z, r) \cap P = \{z\}$. Show that a (non-empty) closed set P with no isolated points (this is also called a *perfect* set) must be uncountable.

3. Problems 2.2.2, 3, 4, 10, 11

"due:" 9/16