# WEYL THEORY

CHRISTIAN REMLING

ABSTRACT. This is an attempt to advertise the use of matrix notation for linear fractional transformations and of (pseudo)hyperbolic distance in the context of the theory of Titchmarsh-Weyl $m$ functions for second order operators.

My goal is to convince the reader that the notation that will be discussed below is a good choice in the context of the theory of Titchmarsh-Weyl $m$ functions. I will present the bare minimum of material needed to make this case. For example, I work in the discrete setting exclusively, although an analogous treatment can be given for continuous operators. I should also make it clear from the beginning that nothing in this note is new.

Consider a Jacobi difference equation

$$a(n)y(n+1) + a(n-1)y(n-1) + b(n)y(n) = zy(n). \tag{1.1}$$

There is a whole zoo of Titchmarsh-Weyl $m$ functions associated with (1.1), so there can't be one universal simple definition. The following recipe seems quite comprehensive, though. Let $f(n,z)$ be a solution of (1.1) and specify the value of the quotient

$$M(n,z) = -\frac{f(n+1,z)}{a(n)f(n,z)}$$

at some $n$. Then evolve according to (1.1) to obtain $m$ functions.

For example, if $M(N,z) = q \in \mathbb{R}$, then $m(z) = M(0,z)$ is the $m$ function of the problem on $\{1,\ldots,N\}$ with boundary condition $q$ at $N$.

A *linear fractional transformation* is a map of the form

$$z \mapsto \frac{az+b}{cz+d},$$

with $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$. These can be handled very conveniently using matrix notation, as follows: define

$$Sz = \frac{az+b}{cz+d}, \qquad S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This notation has a natural interpretation: Identify $z \in \mathbb{C} \subset \mathbb{CP}^1$ with its homogeneous coordinates $z = [z:1]$ and apply the matrix $S$ to the vector $(z,1)^t$ whose components are these homogeneous coordinates. The image vector $S(z,1)^t$ then tells us what the homogeneous coordinates of the image of $z$ under the linear fractional transformation are. In particular, this recipe can also be used to describe the action of $S$ on the whole Riemann sphere $\mathbb{C}_\infty \cong \mathbb{CP}^1$.

*Date*: June 12, 2007.

These remarks also show that the matrix product corresponds to the composition. Put differently, the association

$$S \mapsto \text{linear fractional transformation}$$

is a group homomorphism between $GL(2,\mathbb{C})$ and the non-constant linear fractional transformations.

The formalism immediately yields the familiar geometry of nested disks. Indeed, note that by (1.1), the matrix $S$ that updates the vector $(f(n+1,z), -a(n)f(n,z))^t$ is given by

$$\begin{pmatrix} f(n+1,z) \\ -a(n)f(n,z) \end{pmatrix} = S(a(n), z-b(n)) \begin{pmatrix} f(n,z) \\ -a(n-1)f(n-1,z) \end{pmatrix},$$

$$S(a,w) \equiv \begin{pmatrix} w/a & 1/a \\ -a & 0 \end{pmatrix}. \tag{1.2}$$

Thus we also have that $M(n,z) = S(a(n), z-b(n))M(n-1,z)$. We abbreviate $S_n(z) \equiv S(a(n), z-b(n))$ and $T_n(z) = S_n(z)^{-1}$, and, for $z \in \mathbb{C}^+$, we introduce

$$\mathcal{D}_n(z) = \left\{ T_1(z)T_2(z)\cdots T_n(z)w : w \in \overline{\mathbb{C}^+} \right\}. \tag{1.3}$$

Here, the closure of the upper half plane is taken in the Riemann sphere, so it includes the point $\infty$. The linear fractional transformations $T_j(z)$ map $\mathbb{C}^+$ into itself if $z \in \mathbb{C}^+$. This follows conveniently from formula (1.6) below. Since $\partial\mathcal{D}_n(z)$ is the image of $\mathbb{R} \cup \{\infty\}$ under a linear fractional transformation, it follows that $\mathcal{D}_n(z)$ is a disk which is contained in $\mathbb{C}^+$. Moreover, it is obvious from (1.3) that $\mathcal{D}_{n+1}(z) \subset \mathcal{D}_n(z)$, so these disks are indeed nested.

Finally, (1.3) also provides an interpretation of $\mathcal{D}_n(z)$: The interior of this disk is exactly the collection of those values that $m(z) = M(0,z)$ can take if we know the coefficients $a(j)$, $b(j)$ on the interval $1 \le j \le n$.

As is well known, an immediate consequence of this geometry is the fact that as $n \to \infty$, the disks $\mathcal{D}_n(z)$ approach a limiting object, which must be either a point or a circle. Which of these alternatives holds is independent of $z \in \mathbb{C}^+$ for a given Jacobi operator.

**Theorem 1.1.** *Let $q : \mathbb{C}^+ \to \overline{\mathbb{C}^+}$ be a holomorphic function. Assume limit point case, and let $M(z)$ be the unique limit point of the $\mathcal{D}_n(z)$. Then*

$$T_1(z)T_2(z)\cdots T_n(z)q(z) \to M(z), \tag{1.4}$$

*as $n \to \infty$, for all $z \in \mathbb{C}^+$ (and in fact locally uniformly on $\mathbb{C}^+$).*

This is of course obvious from the preceding discussion, except perhaps for the final claim, which, for example, follows from a normal families argument. The function $q$ plays the role of a choice function; it picks one point from every Weyl disk $\mathcal{D}_n(z)$.

Being obvious doesn't prevent the Theorem from also being very useful. The convergence of the Herglotz functions implies the (weak $*$) convergence of the associated measures. For example, the special case $q \equiv i$ gives the following well known result. Write $v(n,z)$ for the solution of (1.1) with the initial values $a(0)v(0,z) = 0$, $v(1,z) = 1$. Then the spectral measure $\rho$ of the half line Jacobi operator can be obtained as the (weak $*$) limit

$$d\rho(t) = \frac{1}{\pi} \lim_{n\to\infty} \frac{dt}{a(n)^2v(n,t)^2 + v(n+1,t)^2}. \tag{1.5}$$

This can now be proved as follows. Introduce the solution $u$ by requiring that $a(0)u(0,z) = 1$, $u(1,z) = 0$. Denote the left-hand side of (1.4) by $M_n(z)$ and note that $f_n(j,z) = u(j,z) - M_n(z)v(j,z)$ solves (1.1) and $-f_n(1,z)/(a(0)f_n(0,z)) = M_n(z)$, and at $j = n$, this quotient equals $q(z) = i$ by the definition of $M_n(z)$. This implies that

$$M_n(z) = \frac{u(n+1,z) + ia(n)u(n,z)}{v(n+1,z) + ia(n)v(n,z)},$$

and another short computation then shows that the measure associated with $M_n$ is given by the right-hand side of (1.5).

A change of boundary condition can also be described very conveniently in this setting. More precisely, if either $m(z) = M(0,z)$ or $m(z)$ is a half line $m$ function (and thus a limit of $m$'s of the first type), then $m_t(z) = A_t m(z)$ with

$$A_t = S^{-1}(a, w-t)S(a,w) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

is the $m$ function for the problem with the modified coefficient $b_t(1) = b(1) + t$. See [2] for more on this.

The *pseudohyperbolic distance* on $\mathbb{C}^+$ is often helpful in this context. It can be used to analyze the shrinking (or expanding) sets that were discussed above more quantitatively. One can work with the following definition:

$$\gamma(w,z) = \left| \frac{w-z}{w-\overline{z}} \right| \qquad (w, z \in \mathbb{C}^+).$$

Then holomorphic self-maps of $\mathbb{C}^+$ (such as the linear fractional transformations $T_j(z)$ from above) are distance decreasing. In particular, automorphisms of $\mathbb{C}^+$ are isometries with respect to $\gamma$. Recall that $S \in \mathrm{Aut}(\mathbb{C}^+)$ precisely if its matrix has real entries and positive determinant.

The inverses of the matrices $S$ from (1.2) have the factorization

$$S(a,w)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \equiv JT_0A. \tag{1.6}$$

Here, $A, J \in \mathrm{Aut}(\mathbb{C}^+)$ are isometries with respect to $\gamma$, and $T_0(w)\zeta = \zeta + w$ is a translation. See [3, Appendix A] for an application of these ideas.

## Further reading

[1] R. Froese, D. Hasler, and W. Spitzer, Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs, *J. Funct. Anal.* **230** (2006), 184–221.

[2] F. Gesztesy and K.A. Makarov, $\mathrm{SL}_2(\mathbb{R})$, exponential representation of Herglotz functions, and spectral averaging, *St. Petersburg Math. J.* **15** (2004), 393–418.

[3] C. Remling, The absolutely continuous spectrum of Jacobi matrices, preprint, electronically available at `www.math.ou.edu/∼cremling/research/preprints.html`.

[4] B. Simon, Orthogonal Polynomials on the Unit Circe, Part 2, Sect. 10.4, American Mathematical Society, Colloquium Publications, Vol. 54, Providence, 2005.

Mathematics Department, University of Oklahoma, Norman, OK 73019
*E-mail address*: `cremling@math.ou.edu`
*URL*: `www.math.ou.edu/∼cremling`