Reflectionless Jacobi matrices

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\[(Ju)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n\]

\[u \in \ell^2(I), I = \mathbb{Z} \text{ or } \mathbb{Z}_+\]

\[J = \begin{pmatrix}
  \cdots & \cdots & \cdots \\
  & a_{-2} & b_{-1} & a_{-1} \\
  & a_{-1} & b_{0} & a_{0} \\
  & a_{0} & b_{1} & a_{1} \\
  & & & & \cdots & \cdots & \cdots 
\end{pmatrix}\]

\[J_+ = \begin{pmatrix}
  b_{0} & a_{0} \\
  a_{0} & b_{1} & a_{1} \\
  a_{1} & b_{2} & a_{2} \\
  & & & & \cdots & \cdots & \cdots 
\end{pmatrix}\]

\[a, b \in \ell^\infty(I), a_n \geq 0, b_n \in \mathbb{R}\]
$J_+$ is self-adjoint on $\ell^2(\mathbb{Z}_+)$ and has simple spectrum (if $a_n > 0$); $\delta_0$ is a cyclic vector. So

$$J_+ \cong \text{multiplication by } t \text{ in } L^2(\mathbb{R}, d\rho),$$

with $d\rho(t) = d\|E(t)\delta_0\|^2$.

Decompose

$$d\rho(t) = f(t)\, dt + d\rho_s(t)$$

and define

$$\Sigma_{ac} = \{ t \in \mathbb{R} : f(t) > 0 \}.$$ 

Note that $\sigma_{ac} = \Sigma_{ac}^{\text{ess}}$. 

A (whole line) Jacobi matrix $J$ is called *reflectionless* on $A \subset \mathbb{R}$ (notation: $J \in \mathcal{R}(A)$) if for all $n \in \mathbb{Z}$,

$$\text{Re } g_n(t) = 0 \quad \text{for almost every } t \in A$$

where $g_n(z) = \langle \delta_n, (J-z)^{-1}\delta_n \rangle$. Equivalently,

$$J \in \mathcal{R}(A) \iff m_+(t) = -\overline{m_-}(t) \quad \text{for almost every } t \in A$$

or

$$J \in \mathcal{R}(A) \iff R(t) = 0 \quad \text{for almost every } t \in A,$$

$$R(t) = -\frac{m_+(t) + \overline{m_-}(t)}{m_+(t) + m_-}(t)$$

Remarks

(1) If \( J \) is periodic, then \( J \in R(\sigma) \).

(2) If \( J \in R(A) \), then \( \Sigma_{ac}(J_{\pm}) \supset A \).

(3) If \( J, J' \in R(A) \), \( |A| > 0 \), and \( (a, b)_n = (a', b')_n \) for \( n < n_0 \) or \( n > n_0 \), then \( J = J' \).
Theorem 1. $\omega(J_+) \subset \mathcal{R}(\Sigma_{ac}(J_+))$

Here,

$$\omega(J_+) = \{\lim S^n_{nj} J_+\},$$

$J' = S^k J$ has coefficients $(a', b')_n = (a, b)_{n+k}$, and the limit is taken with respect to pointwise convergence; or use

$$d(J, J') = \sum_{n \in \mathbb{Z}} 2^{-|n|} \left(|a_n - a'_n| + |b_n - b'_n|\right).$$

In other words, $J' \in \omega(J_+)$ precisely if there exists a subsequence $n_j \to \infty$ so that

$$a'_n = \lim_{j \to \infty} a_{n+n_j}, \quad b'_n = \lim_{j \to \infty} b_{n+n_j} \quad \text{for all } n \in \mathbb{Z}.$$ 

Theorem 1 relies heavily on previous work of Breimesser and Pearson (2003).
Since reflectionless operators are rare, it’s not easy to produce ac spectrum. Come to think of it, indeed there aren’t many examples:

- periodic coefficients

- certain specific examples with almost periodic coefficients

- small perturbations of these
Two consequences of Theorem 1

(1) The Oracle Theorem:
Write $C(M) = [0, M] \times [-M, M]$ (so $(a, b)_n \in C(M)$).

**Theorem 2.** Fix $M > 0$, $A \subset \mathbb{R}$, $|A| > 0$. Then, for every $\epsilon > 0$, there exist $L \geq 0$ and a smooth function (the oracle)

$$\Delta : C(M)^{L+1} \to C(M)$$

such that the following holds:

If $\|J_+\| \leq M$ and $\Sigma_{ac}(J_+) \supset A$, then

$$\left| (a, b)_{n+1} - \Delta ((a, b)_{n-L}, \ldots, (a, b)_n) \right| < \epsilon$$

for all $n \geq n_0 = n_0(J_+)$.  

This is inspired by work of Kotani on ergodic operators ("Kotani theory", 1984 – present).
Sketch of proof.

\[(a, b)_{n+k} \approx (a', b')_k \quad (k \in \mathbb{Z})\]

for some \(J' \in \omega(J_+) \subset \mathcal{R}(A)\) by Theorem 1. So I approximately know a left half line of a \(J' \in \mathcal{R}(A)\) (product topology!). By Fact (3) from above, this approximately determines \(J'\) and, in particular, it approximately determines \((a, b)_{n+1} \approx (a', b')_1\). □
(2) Denisov-Rakhmanov type Theorems:
For a compact set \( K \subset \mathbb{R} \), define

\[
\mathcal{R}_0(K) = \{ J \in \mathcal{R}(K) : \sigma(J) \subset K \}.
\]

Remarks: (1) If \( K \) is essentially closed,

\[
|K \cap (x - h, x + h)| > 0 \quad \text{for all } x \in K, h > 0,
\]

then all \( J \in \mathcal{R}_0(K) \) will in fact satisfy \( \sigma(J) = K \).
(2) \((\mathcal{R}_0(K), d)\) is a compact space itself.

**Theorem 3.** If \( \Sigma_{ac}(J_+) = \sigma_{ess}(J_+) = K \), then

\[
\omega(J_+) \subset \mathcal{R}_0(K).
\]

In other words,

\[
\lim_{n \to \infty} d(S^n J_+, \mathcal{R}_0(K)) = 0.
\]

This is Theorem 1, plus the easy observation that \( \sigma(J') \subset \sigma_{ess}(J_+) \) for all \( J' \in \omega(J_+) \).
It is well known that $\mathcal{R}_0([-2,2])$ consists of the free Jacobi matrix $((a,b) \equiv (1,0))$ only. So we immediately recover

**Corollary 4** (Denisov 2004). If $\Sigma_{ac}(J_+) = \sigma_{ess}(J_+) = [-2,2]$, then

$$\lim_{n \to \infty} a_n = 1, \lim_{n \to \infty} b_n = 0.$$
So it’s interesting to study the spaces $\mathcal{R}_0(K)$; in fact, ideally, we would like to understand the dynamical systems $(\mathcal{R}_0(K), S)$.

- If $K = \bigcup_{n=0}^{N} [c_n, d_n]$ (finite gap set), then $(\mathcal{R}_0(K), S) \cong (\mathbb{T}^N, T_a)$.

- An analogous result holds for certain infinite gap sets (Sodin-Yuditskii, 1997).

- If $K = [-2, 2] \cup \{x_1, \ldots, x_N\}$, then $\mathcal{R}_0(K)$ consists of solitons.
There is a natural representation of $\mathcal{R}_0(K)$ as a fibered space over $\mathcal{T} = \mathbb{T}^N$; $N \in \mathbb{N}_0 \cup \{\infty\}$ is the number of gaps of $K$. Compare Craig (1989), Kotani,...

$$p : \mathcal{R}_0(K) \to \mathcal{T}$$

continuous, surjective; $p^{-1}(\{t\}) = \text{fiber over } t \in \mathcal{T}$

**Theorem 5.** The function

$$\delta(t) = \text{diam } p^{-1}(\{t\})$$

is upper semicontinuous, and $\{t \in \mathcal{T} : \delta(t) = 0\}$ is a dense $G_\delta$ set.

In general, the set $\{\delta > 0\}$ can also easily be dense.

For sets $K$ that are not too thin anywhere (weakly homogeneous), $\delta = 0$ everywhere and thus $\mathcal{R}_0(K) \cong \mathcal{T}$. 
What are the effects of non-trivial fibers (if any)?

**Theorem 6.** $R_0(K)$ is always pathwise connected.

Suppose that $x_0 \in K$ is an isolated point of $K$ (this leads to non-trivial fibers). Let $K_t = K \cup [x_0 - t, x_0 + t]$.

**Theorem 7.** $R_0(K)$ and $R_0(K_t)$ for $t > 0$ are homeomorphic.

**Corollary 8.** Suppose that

$$K = \bigcup_{j=1}^{N+1} [c_j, d_j]$$

is a disjoint union of $N + 1$ compact intervals, some of which may be single points. Then $R_0(K)$ is homeomorphic to $T^N$. 
Open question: Is there any \( K \subset \mathbb{R} \) for which \( \mathcal{R}_0(K) \) is not a torus?

Alexei Poltoratski and I have also looked at the question of how the \( \mathcal{R}_0(K) \) vary with \( K \). We use Hausdorff distance to compare the \( \mathcal{R}_0(K) \). This seems to correspond to

\[
\delta(K, K') = h(K, K') + |K \Delta K'|.
\]

However, we can again handle only certain special types of non-trivial fibers, and part of the question is left open.
**Theorem 9.** Suppose that \( J \in \mathcal{R}([B - 2A, B + 2A]) \).

Then \( a_n \geq A \) for all \( n \in \mathbb{Z} \). If \( a_{n_0} = A \) for a single \( n_0 \in \mathbb{Z} \), then \( a_n = A, b_n = B \) for all \( n \in \mathbb{Z} \).

Recall the Denisov-Rakhmanov Theorem:

**Theorem 10.** (DR) Suppose that:

1. \( \sigma_{ess}(J_+) = [-2, 2] \);
2. \( \Sigma_{ac}(J_+) = [-2, 2] \).

Then \( a_n \to 1, b_n \to 0 \) as \( n \to \infty \).

Both assumptions are necessary. However, if we make the first part of the conclusion (\( a_n \to 1 \)) an assumption, then (2) can be dropped (Damanik, Hundertmark, Killip, Simon, 2003).
Alternatively, assumption (1) can also be dropped in this situation. More is true:

**Theorem 11.** Suppose that $\Sigma_{ac}(J_+) \supset [-2, 2]$. Furthermore, suppose that there exists a subsequence $n_j \to \infty$ with bounded gaps (that is, $\sup(n_{j+1} - n_j) < \infty$) so that $a_{n_j} \to 1$. Then

$$a_n \to 1, b_n \to 0 \quad (n \to \infty).$$

*Proof.* Same as before: let $J \in \omega(J_+)$. Then $J \in \mathcal{R}(-2, 2)$, but also $a_n(J) \equiv 1$. Thus $J = J_0$, the free Jacobi matrix. \qed


Theorem 12. Suppose that $\Sigma_{ac} \supset (B - 2A, B + 2A)$. Then

$$\liminf_{n \to \infty} a_n \geq A.$$  

Note that:
Potential theory $\Rightarrow \liminf (a_1 a_2 \cdots a_n)^{1/n} \geq A$

Theorem 13 (Strong oracle effect of minimal $a$’s). Suppose that $\Sigma_{ac} \supset (B - 2A, B + 2A)$ and $a_{n_j} \to A$. Then there are $L_j \to \infty$ so that

$$\lim_{j \to \infty} \max_{|k - n_j| \leq L_j} (|a_k - A| + |b_k - B|) = 0.$$
A version for arbitrary sets

Recall that

\[ \{J_0\} = \mathcal{R}_0([-2, 2]). \]

**Theorem 14.** Fix a compact set \( K \subset \mathbb{R} \) of positive Lebesgue measure. Then there exists a constant \( A = A(K) > 0 \) such that the following holds:

(a) If \( J \in \mathcal{R}(K) \), then \( a_n \geq A \) for all \( n \in \mathbb{Z} \).

(b) If \( a_{n_0} = A \) for a single \( n_0 \in \mathbb{Z} \), then \( J \in \mathcal{R}_0(K) \).

(Moreover, there are \( J \in \mathcal{R}_0(K) \) with \( a_0 = A \).)
Corollary 15. Let $B \subset \mathbb{R}$ a bounded Borel set of positive Lebesgue measure. Then there exists a constant $A = A(B) > 0$ so that

$$\liminf_{n \to \infty} a_n \geq A$$

for all $J$ with $\Sigma_{ac} \supset B$.

This is a uniform version of the result of Dombrowski and Simon-Spencer that $\liminf a_n > 0$ if $|\Sigma_{ac}| > 0$. 