# Groups With a Character of Large Degree Relative to a Normal Subgroup 

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#### Abstract

Let $N$ be a normal subgroup of a finite group $G$ and $\theta$ be an irreducible character of $N$ which is fixed by the conjugation action of $G$. Let $\chi$ be an irreducible character of $G$ that restricts to a multiple of $\theta$ on $N$. Then $d=\chi(1) / \theta(1)$ is an integer which divides $|G: N|$ and has $|G: N| \geq d^{2}$. We can thus write $|G / N|=d(d+e)$ for a non-negative integer $e$ and ask what can be said about $d$ and $G / N$ for a given $e$.

This is a generalization of a problem considered by Snyder [12] where he takes $d$ to an irreducible character degree of $G$ and writes $|G|=d(d+e)$. Berkovich has shown in [1] that if $e=1$, then $G$ is a sharply 2-transitive group. For $e \neq 1$, Snyder shows in [12] that $d$ is bounded by a function of $e$. This bound is later improved by Isaacs in [7] and then by Durfee and Jensen in [2] and Lewis in [9]. In this more general version of the problem, we will work under the assumption that $G / N$ is solvable. We will show that for $e>0$, if $d>(e-1)^{2}$ then $e$ divides $d$ and $d / e+1$ is a prime power. If in addition, either $d>e^{5}-e,(d / e, e)=1$, or $(d / e+1, e)=1$ then there exist groups $X, Y$ with $N \subseteq X \triangleleft Y \subseteq G$ such that $Y / X$ is a sharply 2 -transitive group of order $(d / e)(d / e+1)$.


## 1 Introduction

This paper is a generalization of a problem originally considered by Snyder in [12]. The problem is as follows. Let $d$ be the degree of an irreducible character of a finite group $G$. Then $d$ divides $|G|$ and $|G| \geq d^{2}$. We therefore can write $|G|=d(d+e)$ for some non-negative integer $e$ and ask what can be said about $G$ given a specific $e$ value. If $e=0$, then the irreducible character of degree $d$ must be the unique irreducible character of $G$ so $G$ must be the trivial group. The case where $e=1$ has been fully classified by Y. Berkovich in Theorem 7 of [1]. He shows that $e=1$ if and only if either $|G|=2$ or $G$ is a 2-transitive Frobenius group. The group of order 2 together with the 2-transitive Frobenius groups are exactly the sharply 2 -transitive groups. We will refer to them as such in order to avoid a case division.

The case where $e=1$ is in fact an anomaly as there is no upper bound on $|G|$ in this case. It was first proved by N. Snyder in [12] that for $e>1$ there is an upper bound on $|G|$ in terms of $e$. The bound found by Snyder was $|G| \leq((2 e)!)^{2}$. Isaacs later proved in [7] that there is a polynomial bound for $|G|$ given by $B e^{6}$, where $B$ is an unknown universal constant. For certain cases, Isaacs relies on work by Larsen, Malle, and Tiep [8] which appeals to the classification of simple groups. Durfee and Jensen improved this bound to $e^{6}-e^{4}$ in [2] and eliminated the need to use the classification of simple groups. They also showed that for $G$ solvable, the bound for $e$ prime and $e$ divisible by two distinct primes can be improved to $e^{4}-e^{3}$. Lewis finished the solvable case by showing in [9] that $|G| \leq e^{4}-e^{3}$ for $e$ a nontrivial prime power and $G$ solvable. This is a sharp bound for $e$ a nontrivial prime power as Isaacs describes in [7], an example where $|G|=e^{4}-e^{3}$ and $e$ is any nontrivial prime power.

This paper addresses a generalization of this problem. We will refer to it as the relative version of the problem as all work will be relative to a fixed normal subgroup. Let $N \triangleleft G$ and fix $\theta$ to be an irreducible character of $N$ which is invariant under the conjugation action of $G$ on the irreducible characters of $N$. If $\chi$ is an irreducible character of $G$ that has $\theta$ as a constituent when restricted to $N$, we get that $d=\chi(1) / \theta(1)$ is a positive integer which behaves like an irreducible character degree of $G / N$. In particular $d$ divides $|G: N|$ and $|G: N| \geq d^{2}$. Thus as in the original version of the problem, we can write $|G: N|=d(d+e)$ where $e$ is a non-negative integer. Note that if we take $N=\{1\}$, then $\chi$ is an arbitrary irreducible character of $G$ and we are in the situation of the original problem. As the relative version of the problem is more complicated, we focus on the case where $G / N$ is solvable.

A natural place to start is to consider what happens when $e=0$. It turns out that the relative version of the problem is very different from the original version when $e=0$. In the original problem, if $e=0$, then $G$ must be the trivial group so $d=1$. There is no such restriction on $d$ in the relative version of the problem. In fact we will present examples which show that $d$ can be any positive integer when $e=0$. The next case to consider is $e=1$. Unlike the $e=0$ case, the results for $e=1$ in the relative version are similar to those in the original problem. As previously mentioned, Berkovich showed
that $e=1$ if and only if $G$ is a sharply 2 -transitive group. We will show the following more general result for $G / N$ solvable.

Theorem A. Let $N \triangleleft G$ with $G / N$ solvable and let $\theta$ be an irreducible character of $N$ which is invariant under the conjugation action of $G$. Let $\chi$ be an irreducible character of $G$ which has $\theta$ as a constituent when restricted to $N$ and write $|G: N|=d(d+e)$ where $d=\chi(1) / \theta(1)$ and $e$ is a nonnegative integer. If $e=1$, then $G / N$ is a sharply 2-transitive group.

The similarity between the relative and original problems when $e=1$ suggests that there might be a chance that the results are similar for larger $e$ values. One might hope that for $e>1$ we could bound $d$ (and hence $|G: N|$ ) in terms of $e$, as was the case in the original problem. This does not turn out to be true and we will present examples which demonstrate that $d$ is not bounded for any $e$. More specifically, if $d$ and $e$ are positive integers and $d / e+1$ is a prime power, then there exists examples with the given $d$ and $e$ values. Clearly given any $e$ there is no bound on the values of $d$ which make $d / e+1$ a prime power so it is not possible to bound $d$ in terms of $e$. Choosing $d / e+1$ to be a prime power was not arbitrary. In fact when $d$ gets large compared to $e$, this is a necessary condition as stated in the following theorem.

Theorem B. Let $N \triangleleft G$ with $G / N$ solvable and let $\theta$ be an irreducible character of $N$ which is invariant under the conjugation action of $G$. Let $\chi$ be an irreducible character of $G$ which has $\theta$ as a constituent when restricted to $N$ and write $|G: N|=d(d+e)$ where $d=\chi(1) / \theta(1)$ and $e$ is a nonnegative integer. For $e>0$, if $d>(e-1)^{2}$, then $e$ divides $d$ and $d / e+1$ is a prime power.

Thus for a fixed $e$ value we may not be able to bound $d$ in terms of $e$, but we have found some restrictive conditions on possible $d$ values. In addition to these results about $d$, there are things that can be said about $G / N$. The sharply 2 -transitive groups mentioned for $e=1$ will actually appear for larger $e$ values as well. When $d$ gets large, we will see that somewhere between $N$ and $G$ there must be a sharply 2-transitive group of order $(d / e)(d / e+1)$. The following theorem explains in more detail what is meant by "between $N$ and $G$ ".

Theorem C. Let $N \triangleleft G$ with $G / N$ solvable and let $\theta$ be an irreducible character of $N$ which is invariant under the conjugation action of $G$. Let $\chi$ be an irreducible character of $G$ which has $\theta$ as a constituent when restricted to $N$ and write $|G: N|=d(d+e)$ where $d=\chi(1) / \theta(1)$ and $e$ is a nonnegative integer. Assume that $e>0$ and $d>(e-1)^{2}$. If $(d / e, e)=1,(d / e+1, e)=1$, or $d>e^{5}-e$, then there exists groups $X, Y$ with $N \subseteq X \triangleleft Y \subseteq G$ such that $Y / X$ is a sharply 2-transitive group of order $(d / e)(d / e+1)$.

Note that if $e$ is a prime power, then $e$ must always be prime to either $d / e$ or $d / e+1$ as $(d / e, d / e+1)=1$. Thus when $e$ is a prime power, $d$ only needs to be larger than $(e-1)^{2}$ for the theorem to hold.

In addition to these results, we will consider the case when $G / N$ is nilpotent and prove that $|G: N|$ is bounded in this case.

Theorem D. Let $N \triangleleft G$ with $G / N$ nilpotent and let $\theta$ be an irreducible character of $N$ which is invariant under the conjugation action of $G$. Let $\chi$ be an irreducible character of $G$ which has $\theta$ as a constituent when restricted to $N$ and write $|G: N|=d(d+e)$ where $d=\chi(1) / \theta(1)$ and $e$ is a nonnegative integer. If $e>0$, then $d \leq e$ and hence $|G: N| \leq 2 e^{2}$.

There are still some unanswered questions. Testing using the Small Groups Database in MAGMA suggests that the result in Theorem C may actually be that $G / N$ has a subgroup which is a sharply 2-transitive group of order $(d / e)(d / e+1)$. A possible avenue for future research would be to determine if this is always true or to find counterexamples. Another possibility is to consider what happens if we drop the assumption that $G / N$ is solvable.

## 2 Notation and Statement of the Problem

We will start by introducing some notation and assumptions that will be used throughout. All groups will be assumed to be finite. Let $N$ be a normal subgroup of a group $G$. We will write $\operatorname{Irr}(N)$ for the irreducible characters of $N$. Given $\theta \in \operatorname{Irr}(N)$, the set $\operatorname{Irr}(G \mid \theta)$ will denote the irreducible characters $\chi$ of $G$ whose restriction $\chi_{N}$ to $N$ has $\theta$ as a constituent. By Frobenius reciprocity, these are exactly the irreducible constituents of $\theta^{G}$.

As $N \triangleleft G$, there is a natural conjugation action of $G$ on $\operatorname{Irr}(N)$ given by $\theta^{g}(x)=$ $\theta\left(g x g^{-1}\right)$ for $\theta \in \operatorname{Irr}(N), g \in G, x \in N$. We will be restricting our focus to the case where $\theta$ is fixed by the conjugation action of $G$ on $\operatorname{Irr}(N)$, i.e. when $\theta$ is a $G$-invariant character.

Lemma 2.1. Let $N \triangleleft G$ and let $\theta \in \operatorname{Irr}(N)$ be $G$-invariant. Write $d_{\chi}=\chi(1) / \theta(1)$ for $\chi \in \operatorname{Irr}(G \mid \theta)$. Then the $d_{\chi}$ are positive integers which divide $|G: N|$ and satisfy $|G: N|=\sum_{\chi \in \operatorname{Irr}(G \mid \theta)}\left(d_{\chi}\right)^{2}$.

Proof. Let $\chi \in \operatorname{Irr}(G \mid \theta)$. By Theorem 6.2 of [5], the irreducible constituents of $\chi_{N}$ are exactly the $G$-conjugates of $\theta$. As $\theta$ is assumed to be $G$-invariant, $\chi_{N}$ must be a positive integer multiple of $\theta$ so $\chi_{N}=c \theta$ for some positive integer $c$. Considering degrees gives that $c=\chi(1) / \theta(1)=d_{\chi}$ so $d_{\chi}$ is a positive integer.

The fact that $d_{\chi}$ divides $|G: N|$ is proved in Corollary 11.29 of [5].
By Frobenius reciprocity, $d_{\chi}=\left[\chi_{N}, \theta\right]=\left[\chi, \theta^{G}\right]$ and the characters of $\operatorname{Irr}(G \mid \theta)$ are exactly the irreducible constituents of $\theta^{G}$. It follows that $\theta^{G}=\sum_{\chi \in \operatorname{Irr}(G \mid \theta)} d_{\chi} \chi$. Evaluating this expression at 1 gives that $\theta(1)|G: N|=\sum_{\chi \in \operatorname{Irr}(G \mid \theta)} d_{\chi} \chi(1)$ and dividing by $\theta(1)$ we get $|G: N|=\sum_{\chi \in \operatorname{Irr}(G \mid \theta)}\left(d_{\chi}\right)^{2}$.

Throughout this paper we will be working with the hypotheses that $N \triangleleft G$ and $\theta \in \operatorname{Irr}(N)$ is $G$-invariant. For simplicity we will abbreviate these hypotheses by saying that $(G, N, \theta)$ is a character triple. Note that this is the same notation used in Chapter 11 of Isaacs' Character Theory of Finite Groups [5].

Let $(G, N, \theta)$ be a character triple. The integers $d_{\chi}$ mentioned in the previous lemma depend exactly on $G, N$ and $\theta$ and can thus be associated to the character triple $(G, N, \theta)$. We see from Lemma 2.1 that these positive integers satisfy properties which are similar to those of the character degrees of $G / N$. We will therefore refer to these integers as the relative character degrees of $(G, N, \theta)$.

We now have all the notation needed to set-up the problem.
Corollary 2.2. Let $(G, N, \theta)$ be a character triple and d be a relative character degree. Then $|G: N|=d(d+e)$ for some non-negative integer $e$.

Proof. This follows from Lemma 2.1 as $d$ divides $|G: N|$ and $|G: N| \geq d^{2}$.
Our goal will be to determine what can be said about $G / N$ for a fixed $e$. That is, if the character triple $(G, N, \theta)$ has a relative character degree of $d$ with $|G: N|=d(d+e)$. Note that if we take $N=1$, then $\theta=1_{N}$ and the relative character degrees are the irreducible character degrees of $G$ so in this case we get the original problem mentioned in the introduction.

We will start by defining some terminology. If $(G, N, \theta)$ is a character triple and $|\operatorname{Irr}(G \mid \theta)|=1$, then we say that $\theta$ is fully ramified in $G$. There are a number of conditions which are equivalent to a character triple having a fully ramified character as we will show in the next lemma. In the lemma and throughout this paper, we refer to the vanishing-off subgroup of a character. If $\chi$ is a character of $G$, we define the vanishingoff subgroup of $\chi$ to be the subgroup of $G$ generated by the elements $g \in G$ for which $\chi(g) \neq 0$. We denote this group by $\mathbf{V}(\chi)$ and note that $\mathbf{V}(\chi) \triangleleft G$ as $\chi$ is constant on conjugacy classes.

Lemma 2.3. The following are equivalent for the character triple $(G, N, \theta)$.
(1) $\theta$ is fully ramified in $G$
(2) $|\operatorname{Irr}(G \mid \theta)|=1$
(3) $|G: N|=d^{2}$ where $d=\chi(1) / \theta(1)$ for some $\chi \in \operatorname{Irr}(G \mid \theta)$
(4) $\mathbf{V}(\chi) \subseteq N$ for some $\chi \in \operatorname{Irr}(G \mid \theta)$

Proof. Conditions (1) and (2) are equivalent by definition. By Lemma 2.1, we know that

$$
|G: N|=\sum_{\psi \in \operatorname{Irr}(G \mid \theta)}\left(d_{\psi}\right)^{2}
$$

where $d_{\psi}=\psi(1) / \theta(1)$ are positive integers. It follows that $|G: N|=\left(d_{\chi}\right)^{2}$ for some $\chi \in \operatorname{Irr}(G \mid \theta)$ if and only if $\chi$ is the unique member of $\operatorname{Irr}(G \mid \theta)$. Thus (3) is equivalent
to (1) and (2). If $|\operatorname{Irr}(G \mid \theta)|=1$ and $\chi$ is the unique member of $\operatorname{Irr}(G \mid \theta)$, then $\theta^{G}$ is a multiple of $\chi$. As $\theta^{G}$ is zero on $G-N$ we must also have that $\chi$ is zero on $G-N$ so $\mathbf{V}(\chi) \subseteq N$. Thus (2) implies (4). Conversely, if $\chi \in \operatorname{Irr}(G \mid \theta)$ has $\mathbf{V}(\chi) \subseteq N$, then by Lemma 2.29 of [5] we have $\left[\chi_{N}, \chi_{N}\right]=|G: N|[\chi, \chi]=|G: N|$. As $\theta$ is invariant in $G$ we have that $\chi_{N}=d \theta$ where $d=\chi(1) / \theta(1)$. Thus $|G: N|=\left[\chi_{N}, \chi_{N}\right]=[d \theta, d \theta]=d^{2}$ so (4) implies (3) and all four conditions are equivalent.

Note that condition (3) is exactly the situation that $e=0$. In the original version of the problem, if $e=0$, then $G$ had to be the trivial group and $d=1$. This is not the case in the relative version as we will show in the next section.

## 3 Examples

In this section we will present examples which show that $d$ is not bounded for any $e$. For $e=0$, we will show that $d$ can be any positive integer. For $e>0$, if $d$ is any positive integer such that $d / e+1$ is a prime power, then we will build an example of a character triple $(G, N, \theta)$ with relative character degree $d$ and $|G: N|=d(d+e)$.

We will be building most of our examples by taking direct products of groups. Note that if $G=H \times K$, then the irreducible characters of $G$ are exactly of the form $\varphi \times \theta$ where $\varphi \in \operatorname{Irr}(H), \theta \in \operatorname{Irr}(K)$ and $\varphi \times \theta((h, k))=\varphi(h) \theta(k)$ for $(h, k) \in H \times K$. Given any two character triples $\left(G_{1}, N_{1}, \theta_{1}\right)$ and $\left(G_{2}, N_{2}, \theta_{2}\right)$, we have that $N_{1} \times N_{2} \triangleleft G_{1} \times G_{2}$ and $\theta_{1} \times \theta_{2} \in \operatorname{Irr}\left(N_{1} \times N_{2}\right)$ is invariant $G_{1} \times G_{2}$. We can therefore define the direct product of the character triples $\left(G_{1}, N_{1}, \theta_{1}\right)$ and $\left(G_{2}, N_{2}, \theta_{2}\right)$ to be the character triple $\left(G_{1} \times G_{2}, N_{1} \times N_{2}, \theta_{1} \times \theta_{2}\right)$. Note that the members of $\operatorname{Irr}\left(G_{1} \times G_{2} \mid \theta_{1} \times \theta_{2}\right)$ are of the form $\chi_{1} \times \chi_{2}$ where $\chi_{1} \in \operatorname{Irr}\left(G_{1} \mid \theta_{1}\right)$ and $\chi_{2} \in \operatorname{Irr}\left(G_{2} \mid \theta_{2}\right)$

Lemma 3.1. Let $\left(G_{1}, N_{1}, \theta_{1}\right)$ and $\left(G_{2}, N_{2}, \theta_{2}\right)$ be character triples with $\chi_{1} \in \operatorname{Irr}\left(G_{1} \mid \theta_{1}\right)$ and $\chi_{2} \in \operatorname{Irr}\left(G_{2} \mid \theta_{1}\right)$. For $i=1,2$, write $d_{i}=\chi_{i}(1) / \theta_{i}(1)$ and $\left|G_{i}: N_{i}\right|=d_{i}\left(d_{i}+e_{i}\right)$ for some $e_{i}$ non-negative integers. If $(G, N, \theta)$ is the direct product of these character triples, then $\chi=\chi_{1} \times \chi_{2} \in \operatorname{Irr}(G \mid \theta)$ has relative degree $d=d_{1} d_{2}$ and if we write $|G: N|=d(d+e)$, then $e=e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}$

Proof. It is easy to see that

$$
d=\frac{\chi(1)}{\theta(1)}=\frac{\chi_{1}(1) \chi_{2}(1)}{\theta_{1}(1) \theta_{2}(1)}=d_{1} d_{2}
$$

To find $e$, we note that $|G: N|=\left|G_{1}: N_{1}\right|\left|G_{2}: N_{2}\right|$. We can thus solve the equation

$$
\begin{gathered}
d(d+e)=|G: N|=d_{1}\left(d_{1}+e_{1}\right) d_{2}\left(d_{2}+e_{2}\right) \\
=d_{1} d_{2}\left(d_{1} d_{2}+e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}\right)=d\left(d+e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}\right)
\end{gathered}
$$

for $e$ to get $e=e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}$.

Our examples for $e=0$ will be direct products of extra special $p$-groups. A $p$-group $P$ is extra special if its center $Z$ has order $p$ and $P / Z$ is elementary abelian. For any prime $p$ and positive integer $n$, there exists two (up to isomorphism) extra special $p$-groups of order $p^{2 n+1}$. If $P$ is an extra special $p$-group of order $p^{2 n+1}$ with center $Z$, then $Z=P^{\prime}$ and the irreducible characters of $P$ consist of the $p^{2 n}$ linear characters of $P / Z$ and $p-1$ nonlinear characters of degree $p^{n}$. It follows that if $\theta$ is any non-principal character of $Z$ and $\chi \in \operatorname{Irr}(G \mid \theta)$, then $\chi(1)=p^{n}$. As $|P: Z|=p^{2 n}$, we see that $\theta$ must be fully ramified in $P$ and $(P, Z, \theta)$ is a character triple with $d=p^{n}$ and $e=0$.

Theorem 3.2. Given any positive integer d, there exists a character triple ( $G, N, \theta$ ) such that $\theta$ is fully ramified in $G$ and $|G: N|=d^{2}$.

Proof. If $d=1$, then $(1,1,1)$ is a fully ramified triple with relative degree 1 . We can thus assume $d>1$. By Lemma 3.1, we see that if we have two character triples which are fully ramified, then their direct product is also fully ramified and the relative character degrees are multiplied. We can already find fully ramified triples with relative degree $p^{n}$ for any nontrivial positive prime power using extra special $p$-groups. By taking direct products of extra special $p$-groups, we can get fully ramified triples with $d$ any positive integer.

In addition to extra special $p$-groups, we will also be using sharply 2-transitive groups to build examples. If $F$ is a finite field of order $p^{n}$, then the multiplicative group $F^{\times}$acts on the additive group $F^{+}$by multiplication and the resulting semidirect product group is a sharply 2 -transitive group of order $p^{n}\left(p^{n}-1\right)$. Hence sharply 2 -transitive groups of order $p^{n}\left(p^{n}-1\right)$ exist for any prime $p$ and integer $n \geq 0$.

Theorem 3.3. If $d, e$ are any two positive integers such that $d / e+1=p^{n}$ is a prime power, then there exists a character triple $(G, N, \theta)$ with relative character degree $d$ and $|G: N|=d(d+e)$.

Proof. Note that as $d, e$ are positive integers we have that $d / e+1=p^{n}>1$ is a non-trivial prime power. Let $G_{1}$ be a sharply 2 -transitive group of order $p^{n}\left(p^{n}-1\right)$. By Berkovich's theorem [1] there exists $\chi_{1} \in \operatorname{Irr}\left(G_{1}\right)$ with degree $p^{n}-1$. Then the triple $\left(G_{1}, 1,1\right)$ has relative character degree $d_{1}=p^{n}-1$ and $\left|G_{1}: 1\right|=d_{1}\left(d_{1}+e_{1}\right)$ where $e_{1}=1$.

By the previous theorem, there exist fully ramified character triples with relative degree any positive integer. Take $\left(G_{2}, N_{2}, \theta_{2}\right)$ to be such a triple with $\chi_{2}$ the unique member of $\operatorname{Irr}\left(G_{2} \mid \theta_{2}\right)$ and relative character degree $e$. Then $\left|G_{2}: N_{2}\right|=d_{2}\left(d_{2}+e_{2}\right)$ with $d_{2}=e, e_{2}=0$.

Let $(G, N, \theta)$ be the direct product of the two triples and $\chi=\chi_{1} \times \chi_{2}$. By Lemma 3.1, the relative degree associated with $\chi$ is $d_{3}=d_{1} d_{2}=\left(p^{n}-1\right) e=d$ and if we write $|G: N|=d_{3}\left(d_{3}+e_{3}\right)$, we get that $e_{3}=e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}=e$.

For a fixed $e$, there is clearly no bound on the $d$ values for which $d / e+1$ is a prime power so there is no bound on $d$ in terms of $e$. There are however conditions on the possible $d$ values. We will show later if $d>(e-1)^{2}$, then $d / e+1$ must be a prime
power. Note also that the example given had a sharply 2 -transitive group of order $p^{n}\left(p^{n}-1\right)=(d / e+1)(d / e)$ as a direct factor. We will show that under certain conditions, there will be a sharply 2 -transitive group of order $(d / e+1)(d / e)$ between $N$ and $G$, although not necessarily as a direct factor.

## 4 Basic Results and Set-up

Before considering what happens when $e=1$, we will prove some basic results and explain the Set-up that we will be using in many of our proofs. We will often consider how characters induce and restrict along $G$-chief factors. This is well understood if these chief factors are abelian, and thus to simplify the problem we will work under the assumption that $G / N$ is solvable. Note that the example we built in the last section was a direct product of $p$-groups and a sharply 2 -transitive group. If we take the sharply 2-transitive group in our example to be $F^{+} \rtimes F^{\times}$for a finite field $F$, then the example will be solvable. Hence the assumption that $G / N$ is solvable does not change the fact that $d$ is not bounded for any $e$.

We begin by mentioning the possibilities for $\theta^{M}$. The following lemma combines two known results which appear as problems in Chapter 6 of Isaacs' Character Theory of Finite Groups [5].

Lemma 4.1. Let $(G, N, \theta)$ be a character triple. Let $M / N$ be an abelian minimal normal subgroup of $G / N$ and $\varphi \in \operatorname{Irr}(M \mid \theta)$. Then $\theta$ induces to $M$ in one of the following ways.
(a) $\theta$ is fully ramified in $M$ so $\theta^{M}=q \varphi$ and $\varphi_{N}=q \theta$ where $|M: N|=q^{2}$.
(b) $\theta^{M}=\sum_{\lambda \in \operatorname{Irr}(M / N)} \lambda \varphi$ where the $\lambda \varphi \in \operatorname{Irr}(M)$ are distinct extensions of $\theta$.

Proof. By problem 6.12 of [5], if $\theta \in \operatorname{Irr}(N)$ and $M / N$ is an abelian $G$-chief factor, then there are three possibilities for $\theta^{M}$. Only two of these are possible for $\theta$ invariant in $G$. Either $\theta$ is fully ramified in $M$ (case a) or $\theta^{M}$ is the sum of $|M: N|$ distinct extensions of $\theta$ (case b). By problem 6.2, the members of $\operatorname{Irr}(M \mid \theta)$ form a single orbit under the multiplicative action of $\operatorname{Irr}(M / N)$ on $\operatorname{Irr}(M)$. It follows that if $\theta$ extends to $\varphi \in \operatorname{Irr}(M)$, then the members of $\operatorname{Irr}(M \mid \theta)$ have the form $\lambda \varphi$ for $\lambda \in \operatorname{Irr}(M / N)$. As $|\operatorname{Irr}(M \mid \theta)|=|M: N|=|\operatorname{Irr}(M / N)|$, these extensions are distinct.

We will now give a definition and set up the notation that we will need in order to apply an inductive argument.

Definition 4.2. Let $M \triangleleft G, \varphi \in \operatorname{Irr}(M)$, and $T=\operatorname{Stab}_{G}(\varphi)$. If $\chi \in \operatorname{Irr}(G \mid \varphi)$, then there exists a unique character $\eta \in \operatorname{Irr}(T \mid \varphi)$ such that $\eta^{G}=\chi$ (see Theorem 6.11 of [5]). We say that $\eta$ is the Clifford correspondent for $\varphi$ and $\chi$.

Set-up 4.3. Let $(G, N, \theta)$ be a character triple with $G / N$ solvable. Let $\chi \in \operatorname{Irr}(G \mid \theta)$ and $d=\chi(1) / \theta(1)$ be the relative character degree associated with $\chi$. Write $|G: N|=d(d+e)$ for some non-negative integer $e$. Let $M / N$ be a minimal normal subgroup of $G / N$. Let $\varphi$ be an irreducible constituent of $\chi_{M}$ and note that $\varphi \in \operatorname{Irr}(M \mid \theta)$. Let $T=\operatorname{Stab}_{G}(\varphi)$, $t=|G: T|$, and $\eta \in \operatorname{Irr}(T)$ be the Clifford correspondent for $\varphi$ and $\chi$. Then $(T, M, \varphi)$ is a character triple with relative character degree $d_{1}=\eta(1) / \varphi(1)=\chi(1) /(t \varphi(1))$ and we can write $|T: M|=d_{1}\left(d_{1}+e_{1}\right)$ for some non-negative integer $e_{1}$.

As $\theta$ is $G$-invariant, $G$ acts on $\operatorname{Irr}(M \mid \theta)$ by conjugation. A useful case division will be to consider whether or not $G$ is transitive on the set $\operatorname{Irr}(M \mid \theta)$.

Lemma 4.4. Assume the notation of Set-up 4.3 and that $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$. Then $d_{1}=d / q$ and $e_{1}=e / q$ for some integer $q>1$.

Proof. We consider the two cases of Lemma 4.1 separately. If $\theta$ is fully ramified in $M$, then $\varphi(1)=q \theta(1)$ where $|M: N|=q^{2}$. Note that $q$ is an integer and $q>1$ as $M / N$ is nontrivial. As $\varphi$ is the unique member of $\operatorname{Irr}(M \mid \theta)$, it must be invariant in $G$ so $T=G$ and $t=1$. Plugging in $t=1$ and $\varphi(1)=q \theta(1)$ to the formula for $d_{1}$ in Set-up 4.3 we get that $d_{1}=\chi(1) / \varphi(1)=\chi(1) /(q \theta(1))=d / q$. Then

$$
d(d+e)=|G: N|=|G: M||M: N|=d_{1}\left(d_{1}+e_{1}\right) q^{2}=(d / q)\left(d / q+e_{1}\right) q^{2}
$$

and solving for $e_{1}$ we get $e_{1}=e / q$.
Next suppose that $\operatorname{Irr}(M \mid \theta)$ consists of $|M: N|$ distinct extensions of $\theta$. Then $\varphi$ is among these extensions so $\varphi(1)=\theta(1)$. Let $q=|M: N|$ and note that $q$ is an integer greater than 1. By assumption, $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$ so the $q$ distinct members of $\operatorname{Irr}(M \mid \theta)$ are an orbit under the conjugation action of $G$. It follows that $t=q$ and $d_{1}=\chi(1) /(t \varphi(1))=\chi(1) /(q \theta(1))=d / q$. As before, we have

$$
\begin{gathered}
d(d+e)=|G: N|=|G: T||T: M||M: N| \\
\quad=t d_{1}\left(d_{1}+e_{1}\right) q=q(d / q)\left(d / q+e_{1}\right) q
\end{gathered}
$$

and solving for $e_{1}$ yields $e_{1}=e / q$.
We next focus on the case where $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$.
Lemma 4.5. Assume the notation of Set-up 4.3 and that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$. Then $d_{1}=d / t$ and $e>0$. Furthermore if $d \geq(e-1)^{2}$, then $e_{1}=0$ and $(d / t)(q-t)=e$ where $q=|M: N|$ and both $(d / t)$ and $(q-t)$ are positive integers.

Proof. Fix $\varphi$ in $\operatorname{Irr}(M \mid \theta)$ to be a constituent of $\chi_{M}$ as in Set-up 4.3. Note that $|\operatorname{Irr}(M \mid \theta)|>$ 1 as $G$ is not transitive on this set so $\theta$ is not fully ramified in $M$. By Lemma $4.1, \varphi$ is an extension of $\theta$ so $d_{1}=\chi(1) /(t \varphi(1))=\chi(1) /(t \theta(1))=d / t$. We have

$$
d(d+e)=|G: N|=|G: T||T: M||M: N|=t(d / t)\left(d / t+e_{1}\right) q .
$$

With some rearranging this becomes $(d / t)(q-t)=e-q e_{1}$. Note that $q-t$ is the number of members of $\operatorname{Irr}(M \mid \theta)$ which are not conjugate to $\varphi$ so it is a positive integer. Also, $d / t=d_{1}$ is a relative character degree so it is a positive integer. It follows that $e-q e_{1}$ is positive so as $q>0$ and $e_{1} \geq 0$ we must have $e>0$.

Suppose $e_{1} \neq 0$. Then $q e_{1}>1$ and $e-q e_{1}$ is a positive integer, divisible by the positive integer $d / t$ so $d / t \leq e-q e_{1}<e-1$. Multiplying by $t$ gives $d<(e-1) t$. We also know that $q-t \geq 1$ so $t \leq q-1$ and $d<(e-1)(q-1)$. Note also that $q<e$ as $e-q e_{1}$ is positive with $e_{1} \neq 0$ so $d<(e-1)^{2}$. It follows that if $d \geq(e-1)^{2}$, then $e_{1}=0$. Plugging in $e_{1}=0$ to the equation $(d / t)(q-t)=e-q e_{1}$ gives us $(d / t)(q-t)=e$.

Lemma 4.6. Let $(G, N, \theta)$ be a character triple with $G / N$ solvable and let $M / N$ be a minimal normal subgroup of $G / N$. Assume that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$. Then there exists $\psi \in \operatorname{Irr}(M \mid \theta)$ which is $G$-invariant. If $\varphi \in \operatorname{Irr}(M \mid \theta)$, then $\varphi=\lambda \psi$ for some $\lambda \in \operatorname{Irr}(M / N)$ such that $\operatorname{Stab}_{G}(\varphi)=\operatorname{Stab}_{G}(\lambda)$.

Furthermore, if $G$ is transitive on $\operatorname{Irr}(M \mid \theta) \backslash\{\psi\}$, then $G$ acts transitively on the nonidentity elements of $M / N$ by conjugation.
Proof. In Lemma 6.1 of [4], Isaacs proves that if $(G, N, \theta)$ is a character triple with $G$ solvable and $M / N$ a minimal normal subgroup of $G / N$, then either $G$ acts transitively on $\operatorname{Irr}(M \mid \theta)$ or a member of $\operatorname{Irr}(M \mid \theta)$ is $G$-invariant. The proof of this lemma can easily be modified to work under the assumption that $G / N$ in solvable (instead of $G$ solvable) as $N$ is in the kernel of all the actions. Then as $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$, there exists $\psi \in \operatorname{Irr}(M \mid \theta)$ which is $G$-invariant.

From Lemma 4.1, we see that the characters $\lambda \psi$ are distinct for $\lambda \in \operatorname{Irr}(M / N)$ and are exactly the members of $\operatorname{Irr}(M \mid \theta)$. It follows that if $\varphi$ is any member of $\operatorname{Irr}(M \mid \theta)$, it must be equal to $\lambda \psi$ for some $\lambda \in \operatorname{Irr}(M \mid \theta)$. As $\psi$ is $G$-invariant it is clear that $\operatorname{Stab}_{G}(\lambda) \subseteq \operatorname{Stab}_{G}(\lambda \psi)$. Conversely, suppose that $g$ fixes $\lambda \psi$. Then $\lambda \psi=(\lambda \psi)^{g}=\lambda^{g} \psi$. The products of $\psi$ with the members of $\operatorname{Irr}(M / N)$ are distinct so $\lambda \psi=\lambda^{g} \psi$ implies that $\lambda^{g}=\lambda$. It follows that $\operatorname{Stab}_{G}(\lambda)=\operatorname{Stab}_{G}(\lambda \psi)=\operatorname{Stab}_{G}(\varphi)$

If $G$ is transitive on $\operatorname{Irr}(M \mid \theta) \backslash\{\psi\}$ and $\varphi \in \operatorname{Irr}(M \mid \theta) \backslash\{\psi\}$, then $\varphi$ is in an orbit of size $|\operatorname{Irr}(M \mid \theta)|-1$ and $\varphi=\lambda \psi$ for some $\lambda \in \operatorname{Irr}(M / N)$. As $\varphi$ and $\lambda$ have the same stabilizer, $\lambda$ is also in an orbit of size $|\operatorname{Irr}(M \mid \theta)|-1=|\operatorname{Irr}(M / N)|-1$. This implies that $G$ is transitive on the non-principal elements of $\operatorname{Irr}(M / N)$ so $\operatorname{Irr}(M / N)$ consists of exactly two $G$-orbits. By Corollary 6.33 of [5], the number of orbits of $G$ acting on $\operatorname{Irr}(M / N)$ by conjugation is the same as the number of orbits of $G$ acting on $M / N$ by conjugation and hence $M / N$ consists of exactly two $G$-orbits. The identity is in its own orbit so $G$ is transitive on the nonidentity elements of $M / N$.

## 5 The $e=1$ Case

We now have enough information to show that if $e=1$, then $G / N$ is a sharply 2-transitive group (for $G / N$ solvable). In practice, we will be describing these groups in terms of a specific normal subgroup.

Lemma 5.1. A solvable group $G$ is a sharply 2-transitive group if and only if $G$ has a normal subgroup $N$ such that $|N|=|G: N|+1$ and $G$ acts transitively on the nonidentity elements of $N$ via conjugation.

Proof. If $G$ acts sharply 2-transitively on a set $\Omega$, then $G$ acts primitively on $\Omega$. As $G$ is a solvable primitive permutation group, $G$ contains a unique minimal normal subgroup $N$ with $|N|=|\Omega|$ which is complemented by a point stabilizer $H$. The action of $H$ on $N$ via conjugation is isomorphic to the action of $H$ on $\Omega$ (see Chapter 1, Section 2 of [10]). It follows from the action being sharply 2-transitive that $|G: N|=|H|=|\Omega|-1=|N|-1$.

Conversely if $G$ has a normal subgroup $N$ satisfying the above hypotheses then there exists a complement $H$ for $N$ in $G$. One can check that the action of $G$ on the right cosets of $H$ by right multiplication is sharply 2 -transitive.

Note that the subgroup $N$ described in the previous lemma must be an elementary abelian $p$-group so these groups have order $p^{n}\left(p^{n}-1\right)$ where $p^{n}$ is a prime power.

We now return to the situation where $e=1$.
Theorem 5.2. Let $(G, N, \theta)$ be a character triple with $G / N$ solvable. Let $\chi \in \operatorname{Irr}(G \mid \theta)$ and let $d=\chi(1) / \theta(1)$ be the relative character degree associated with $\chi$. If $|G: N|=$ $d(d+1)$, then $G / N$ is a sharply 2-transitive group.

Proof. As in Set-up 4.3, take $M / N$ to be a minimal normal subgroup of $G / N$ and $\varphi$ to be a constituent of $\chi_{M}$. If $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$, then Lemma 4.4 implies that $e_{1}=e / q=1 / q$ for some positive integer $q$. This contradicts that $e_{1}$ must be an integer so $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$.

Since $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ and $d>(e-1)^{2}=0$, we conclude by Lemma 4.5 that $e_{1}=0$ and $(d / t)(q-t)=1$ where $q=|M: N|$ and $d / t$ and $q-t$ are positive integers. Thus $q-t=1$ and $d / t=1$ so $t=d$ and $q=t+1=d+1$ so we get that $|G: T|=t=d,|T: M|=(d / t)^{2}=1$, and $|M: N|=q=d+1$. Then as as $q-t=1$, there is exactly one member of $\operatorname{Irr}(M \mid \theta)$ which is not $G$-conjugate to $\varphi$, denote this by $\psi$. Note that $\psi$ is $G$-invariant and $G$ is transitive on $\operatorname{Irr}(M \mid \theta) \backslash\{\psi\}$ so by Lemma 4.6 it follows that $G$ acts transitively by conjugation on the nonidentity elements of $M / N$. Then by Lemma 5.1, $G / N$ is a sharply 2 -transitive group.

Note that Berkovich's result in [1] was an "if and only if" statement. In addition to showing that if $G$ has an irreducible character of degree $d$ with $|G|=d(d+1)$, then $G$ is a sharply 2 -transitive group, he also proved the converse. That is, if $G$ is a sharply 2-transitive group, then $G$ has an irreducible character of degree $d$ where $|G|=d(d+1)$. In the relative case, the converse would be that given any character triple $(G, N, \theta)$ with $G / N$ a sharply 2 -transitive group there exists $\chi \in \operatorname{Irr}(G \mid \theta)$ with relative degree $d$ and $|G: N|=d(d+1)$. This turns out not to be the case. For example, there exists a group $G$ of order 24 with center $Z$ of order 2 and $G / Z$ a sharply 2-transitive group of order 12 for which there is only one irreducible character of degree 3. This character has $Z$ in its kernel so if $\theta \neq 1_{Z} \in \operatorname{Irr}(Z)$, we get that $(G, Z, \theta)$ is a character triple with $G / Z$ sharply 2-transitive of order 12 that does not have 3 as a relative character degree.

## $6 G$ not Transitive on $\operatorname{Irr}(M \mid \theta)$ and $d$ greater than $(e-1)^{2}$

Continuing to assume the Set-up of 4.3, we return to the two cases mentioned in Section 4. Looking at Lemma 4.4, we know that if $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$, then the character triple $(T, M, \varphi)$ has $d_{1}=d / q$ and $e_{1}=e / q$ where $q$ is a positive integer greater than one. This situation will allow us to use induction on $e$ to draw conclusions about $(T, M, \varphi)$. The other case, when $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$, is much more complicated. Lemma 4.5 tells us that if we work under the assumption that $d \geq(e-1)^{2}$, then we have that $e_{1}=0$, or equivalently that $\varphi$ is fully ramified in $T$. In this section, we will show that if we strengthen that assumption to $d>(e-1)^{2}$, then we can prove a lot of things about the structure of $G / N$.

Lemma 6.1. Assume the situation of Set-up 4.3. If $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ and $d \geq(e-1)^{2}$, then $M=N \mathbf{V}(\chi)$.

Proof. By Lemma 4.5, we have that $\varphi$ is fully ramified in $T$ so $\varphi^{T}=f \eta$ for some positive integer $f$. Then as $\eta$ is the Clifford correspondent for $\varphi$ and $\chi$, it follows that $\varphi^{G}=\left(\varphi^{T}\right)^{G}=(f \eta)^{G}=f \chi$. As $M \triangleleft G$, we know that $\varphi^{G}$ vanishes off of $M$ so $\chi$ must as well and $\mathbf{V}(\chi) \subseteq M$. Then $N \mathbf{V}(\chi) \triangleleft G$ and $N \subseteq N \mathbf{V}(\chi) \subseteq M$ with $M / N$ a $G$-chief factor. It follows that $N \mathbf{V}(\chi)$ is either $M$ or $N$. If $N \mathbf{V}(\chi)=N$, then $\mathbf{V}(\chi) \subseteq N$ so Lemma 2.3 implies that $|G: N|=d^{2}$. This is a contradiction as $|G: N|=d(d+e)$ and Lemma 4.5 states that $e>0$. It follows that $N \mathbf{V}(\chi)=M$.

We are now ready to prove the main result of this section.
Lemma 6.2. Assume the situation of Set-up 4.3. If $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ and $d>(e-1)^{2}$, then we have the following:
(a) $|G: T|=d / e$
(b) $|T: M|=e^{2}$
(c) $|M: N|=d / e+1$
(d) $G$ acts transitively on the nonidentity elements of $M / N$

Proof. By Lemma 4.6 there exists $\psi \in \operatorname{Irr}(M \mid \theta)$ which is $G$-invariant and $\lambda \in \operatorname{Irr}(M / N)$ such that $\varphi=\lambda \psi$ and $T$ is also the stabilizer in $G$ of $\lambda$. We begin by proving that $e t=d$.

Assume for contradiction that $e t \neq d$. Referring to Lemma 4.5, we have that $e>0$ and $e=(d / t)(q-t)$ where $q=|M: N|$. Some rearranging gives $e t / d=q-t$. As $|\operatorname{Irr}(M / N)|=q=|\operatorname{Irr}(M \mid \theta)|$, we see that $q-t$ is both the number of members of $\operatorname{Irr}(M \mid \theta)$ which are not conjugate to $\varphi$ and the number of members of $\operatorname{Irr}(M / N)$ which are not conjugate to $\lambda$. Thus $q-t=e t / d$ is a positive integer. We assumed $e t \neq d$ so $e t / d \neq 1$ and hence $e t / d>1$. Note that as $e>0$ and $d$ and $e$ are both non-negative
integers, the assumption that $d>(e-1)^{2}$ implies that $d \geq e$. We therefore cannot have $t=1$ or else the inequality $e t / d>1$ would become $e>d$ which would contradict $d \geq e$.

We will show that every character $\alpha \in \operatorname{Irr}(G / N)$ such that $\alpha_{M}$ does not have $\lambda$ as an irreducible constituent must have $M$ in its kernel. Suppose that $\beta \in \operatorname{Irr}(G \mid \theta)$ is different from $\chi$. By Lemma 4.5, we have that $\varphi$ is fully ramified in $T$ so $\eta$ is the unique member of $\operatorname{Irr}(T \mid \varphi)$. Also $\eta^{G}=\chi$ and thus $\chi$ is the unique member of $\operatorname{Irr}(G \mid \eta)$. It follows that $\chi$ is the only member of $\operatorname{Irr}(G \mid \varphi)$ so $\beta_{M}$ does not have $\varphi$ as an irreducible constituent. The first step towards showing that $\alpha$ has $M$ in its kernel will be to show that $[\alpha \beta, \chi]=0$. To do this, we will show that $(\alpha \beta)_{M}$ has fewer than $t$ irreducible constituents, while $\chi_{M}$ has exactly $t$ irreducible constituents (the $t$ conjugates of $\varphi$ ).

As previously mentioned, $q-t=e t / d$ is both the number of members of $\operatorname{Irr}(M \mid \theta)$ which are not conjugate to $\varphi$ and the number of members of $\operatorname{Irr}(M / N)$ which are not conjugate to $\lambda$. Note also that $\lambda$ and $\varphi$ are in orbits of size $t>1$ so are not invariant in $G$. Hence $\lambda \neq 1_{M / N}$ and $\varphi \neq \psi$. Thus $1_{M / N}$ is among the et/d members of $\operatorname{Irr}(M / N)$ which are not conjugate to $\lambda$. As the principal character is in its own orbit of size one and $e t / d>1$, the largest possible orbit size for any character in $\operatorname{Irr}(M / N)$ which is not conjugate to $\lambda$ is $(e t / d)-1$. Similarly, $\psi$ is in an orbit of size one and is not conjugate to $\varphi$ so the largest possible orbit size for any character in $\operatorname{Irr}(M \mid \theta)$ which is not conjugate to $\varphi$ is $(e t / d)-1$. Note that $M \triangleleft G$ and $\alpha \in \operatorname{Irr}(G)$ so the irreducible constituents of $\alpha_{M}$ form a single $G$-orbit. This orbit is contained $\operatorname{in} \operatorname{Irr}(M / N)$ and is not the orbit containing $\lambda$, thus $\alpha_{M}$ has at most (et/d)-1 irreducible constituents. Similarly, the constituents of $\beta_{M}$ are a $G$-orbit contained in $\operatorname{Irr}(M \mid \theta)$ distinct from the orbit containing $\varphi$. Thus $\beta_{M}$ has at most $(e t / d)-1$ irreducible constituents.

The characters in $\operatorname{Irr}(M / N)$ are linear so the product of a character in $\operatorname{Irr}(M / N)$ with a character in $\operatorname{Irr}(M \mid \theta)$ is irreducible. As $(\alpha \beta)_{M}=\alpha_{M} \beta_{M}$ it follows that the irreducible constituents of $(\alpha \beta)_{M}$ are all of the form $\mu \nu$ where $\mu$ is an irreducible constituent of $\alpha_{M}$ and $\nu$ is an irreducible constituent of $\beta_{M}$. Hence the number of irreducible constituents of $(\alpha \beta)_{M}$ must be less than or equal to the number of irreducible constituents of $\alpha_{M}$ times the number of irreducible constituents of $\beta_{M}$. Thus $(\alpha \beta)_{M}$ has at most $[(e t / d)-1]^{2}$ irreducible constituents. We now wish to show that $[(e / t) / d-1]^{2}<t$. By Lemma 4.5, we know that $d_{1}=d / t$ is a positive integer so $t \leq d$. It follows that

$$
\frac{e t}{d}-1=\frac{e t-d}{d} \leq \frac{e t-t}{d}=\frac{(e-1) t}{d} .
$$

Then as the left hand side of this inequality is positive, we can square both sides to get

$$
\left(\frac{e t}{d}-1\right)^{2} \leq\left(\frac{(e-1) t}{d}\right)^{2}=\left(\frac{t}{d}\right)\left(\frac{(e-1)^{2}}{d}\right) t<t
$$

where the last inequality follows from the fact that $t \leq d$ and $d>(e-1)^{2}$. This implies that $(\alpha \beta)_{M}$ has fewer than $t$ irreducible constituents and as $\chi_{M}$ has $t$ irreducible constituents, $\chi$ cannot be in the irreducible decomposition of $\alpha \beta$ so $[\alpha \beta, \chi]=0$.

Next consider the character $\chi \bar{\alpha}$. As $\bar{\alpha}$ has $N$ in its kernel, we have that $(\chi \bar{\alpha})_{N}$ is a multiple of $\theta$ so the irreducible constituents of $\chi \bar{\alpha}$ must all belong to $\operatorname{Irr}(G \mid \theta)$. We know
that $[\beta, \chi \bar{\alpha}]=[\alpha \beta, \chi]=0$ and $\beta$ was an arbitrary member of $\operatorname{Irr}(G \mid \theta)$ different from $\chi$. It follows that $\chi$ is the unique irreducible constituent of $\chi \bar{\alpha}$. Considering degrees we get that $\chi \bar{\alpha}=\bar{\alpha}(1) \chi$. By Lemma 3.2 of [2] this implies that $\mathbf{V}(\chi) \subseteq \operatorname{ker}(\bar{\alpha})$. We know from Lemma 6.1 that $M=N \mathbf{V}(\chi)$ so we get $M=N \mathbf{V}(\chi) \subseteq \operatorname{ker}(\bar{\alpha})=\operatorname{ker}(\alpha)$.

We have thus proved that every $\alpha \in \operatorname{Irr}(G / N)$ which is not in $\operatorname{Irr}(G \mid \lambda)$ has $M$ in its kernel. Hence $\operatorname{Irr}(G / N)$ is the union of $\operatorname{Irr}(G \mid \lambda)$ and $\operatorname{Irr}\left(G \mid 1_{M}\right)$ so any non-principal character in $\operatorname{Irr}(M / N)$ must be in the same orbit as $\lambda$. Thus $\operatorname{Irr}(M / N)$ consists of exactly the $t$ conjugates of $\lambda$ and $1_{M / N}$ so $q=|\operatorname{Irr}(M / N)|=t+1$. But we know that $q-t=e t / d$ so this forces $e t / d=1$. This contradicts our assumption that $e t \neq d$ so in fact $e t=d$ as desired.

It now remains to show that $e t=d$ implies all of our results. Set-up 4.3 states that $|G: T|=t$ and $|T: M|=d_{1}\left(d_{1}+e_{1}\right)$. Solving $e t=d$ for $t$ we get that $|G: T|=t=d / e$ which is result $(a)$. By Lemma 4.5, as $d>(e-1)^{2}$ we have $d_{1}=d / t$ and $e_{1}=0$ so $|T: M|=(d / t)^{2}=e^{2}$, which is result (b). Solving $q-t=e t / d$ for $q$ we get $|M: N|=q=t+(e t / d)=d / e+1$ which is result $(c)$. Finally, $\varphi \in \operatorname{Irr}(M \mid \theta)$ is in a $G$-orbit of size $t=d / e$ and $|\operatorname{Irr}(M \mid \theta)|=|M / N|=d / e+1$ so $G$ is transitive on $\operatorname{Irr}(M \mid \theta) \backslash\{\psi\}$ and Lemma 4.6 tells us that $G$ is transitive on the nonidentity elements of $M / N$ which is result (d).

## 7 Finding Other Minimal Normal Subgroups of G/N

Recall that the minimal normal subgroup $M / N$ of $G / N$ in Set-up 4.3 was chosen arbitrarily. If we assume that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ and that $d>(e-1)^{2}$, then Lemma 6.1 implies that this intransitivity can occur for at most one choice of a minimal normal subgroup of $G / N$. Thus one possible way to deal with the tricky case where $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ is to replace $M / N$ with another minimal normal subgroup $L / N$. If there is any hope that such an $L$ exists, we will need the centralizer in $G$ of $M / N$ to be strictly larger than $M$. We will start by proving a lemma that shows this must be the case. Note also that the $e=1$ case has already been classified in Theorem 5.2 , so we are primarily concerned with the cases where $e>1$.

Lemma 7.1. Assume the notation of Set-up 4.3 and that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ and $d>(e-1)^{2}$. Let $C$ be the centralizer in $G$ of $M / N$. Then $M \subseteq C \subseteq T$ and if $e>1$, we have $M<C$.

Proof. As $M / N$ is abelian, it is clear that $M \subseteq C$. Note also that as $M / N$ is central in $C$, we have that $C$ fixes all of $\operatorname{Irr}(M / N)$. By Lemma 4.6 there exists $\psi \in \operatorname{Irr}(M \mid \theta)$ which is $G$-invariant and $\varphi=\lambda \psi$ for some $\lambda \in \operatorname{Irr}(M / N)$ and $T$ is also the stabilizer in $G$ of $\lambda$. As $C$ fixes $\lambda$ it follows that $C \subseteq T$. We next aim to show that if $e>1$, then $M<C$.

By Lemma 6.2, we have that $|G: T|=d / e,|T: M|=e^{2},|M: N|=d / e+1$, and $G$ is transitive on the nonidentity elements of $M / N$. We start by showing that $T / M$ is not a cyclic group. By Lemma 4.5, we know that $\varphi \in \operatorname{Irr}(M \mid \theta)$ is fully ramified in $T$ with
$|T: M|=e^{2}>1$ so $\varphi$ does not extend to $T$. If $T / M$ were cyclic, Corollary 11.22 of [5] would imply that $\varphi$ extends to $T$, and as this is not the case $T / M$ is not cyclic.

The group $M / N$ is an abelian minimal normal group so it is elementary abelian of order $p^{n}$ for some prime $p$. Also, $\operatorname{Irr}(M / N) \cong M / N$ so we can think of $\operatorname{Irr}(M / N)$ as a vector space which we will denote $V$. By Corollary 6.33 of [5], the number of orbits a group acting on $M / N$ will equal the number of orbits of the group acting on $\operatorname{Irr}(M / N)$. It follows that the centralizers in $G$ of $\operatorname{Irr}(M / N)$ and $M / N$ are equal so $C$ is the kernel of the action of $G$ on $V$ and $G / C$ acts faithfully on $V$.

Also by Lemma 4.6, $T$ is the stabilizer of a point $\lambda \in \operatorname{Irr}(M / N)$. If $\lambda=1_{M / N}$, then $T=G$ so $d / e=1, M / N=2, C=T=G$ and $|C: M|=(d / e) e^{2}=e^{2}>1$. We are done in that case so we can assume $\lambda \neq 1_{M / N}$. Then as $\lambda$ is in an orbit of size $d / e=|\operatorname{Irr}(M / N)|-1$, we see that $G$ is transitive on the nonidentity elements of $V$. Thus $G / C$ acts faithfully on $V$ and is transitive on the nonzero vectors of $V$. We will next show that this implies that either $T / C$ is cyclic or $G / C \cong \mathrm{GL}_{2}(3)$ and $T / C \cong S_{3}$.

As $G / C$ is solvable and acts faithfully on $V$ and transitively on the nonzero vectors of $V, G / C$ is either isomorphically contained in the group of semilinear transformations of $V$ or $p^{n}=3^{2}, 5^{2}, 7^{2}, 11^{2}, 23^{2}, 3^{4}$ (see Theorem 19.9 of [11] or Theorem 6.8 of [10]). The group of semilinear transformations of $V$, which we will denote $T\left(p^{n}\right)$, is the set of maps from $V$ to $V$ of the form $x \mapsto a x^{\sigma}$ where $a$ is a nonzero element of the field $\mathbb{F}_{p^{n}}$ and $\sigma$ an element of the $\operatorname{Gal}\left(\mathbb{F}_{p^{n}} / \mathbb{F}_{p}\right)$. If we fix $\sigma$ to be the identity, we get a cyclic normal subgroup of order $p^{n}-1$ which we will denote as $L$. The factor group $T\left(p^{n}\right) / L$ is isomorphic to $\operatorname{Gal}\left(\mathbb{F}_{p^{n}} / \mathbb{F}_{p}\right)$ and is thus a cyclic group of order $n$. No nonidentity element of $L$ fixes any element of $V$ so the nontrivial point stabilizers in $T\left(p^{n}\right)$ intersect $L$ trivially. These stabilizers have order $n$ and are thus isomorphic to the cyclic group $T\left(p^{n}\right) / L$. If $G / C$ is isomorphically contained in $T\left(p^{n}\right)$ then $T / C$ is isomorphically contained in a nontrivial point stabilizer of $T\left(p^{n}\right)$ so $T / C$ is cyclic.

We now consider what happens in the exceptional cases. From Theorem 6.8 of [10], we can determine the orders of all possible exceptional cases and dividing by $p^{n}-1$ gives the order of $T / C$. In most of these cases, $|T / C|$ is prime so it is clear that $T / C$ is cyclic. One case where $|T / C|$ is not prime is when $p^{n}=3^{2}$ and $|G / C|=48$. Here $G / C$ must be all of $\mathrm{GL}_{2}(3)$, so $T / C \cong S_{3}$. Another case where $|T / C|$ is not prime is when $p^{n}=5^{2}$ and $|T / C|=4$. The nontrivial point stabilizers in $\mathrm{GL}_{2}(5)$ have order 20 and hence the Sylow 2-subgroups are order 4 so $T / C$ is isomorphic to a Sylow 2-subgroup of a point stabilizer. These subgroups are cyclic as $\left[\begin{array}{ll}1 & 0 \\ 0 & 2\end{array}\right]$ is a an element of order 4 which stabilizes a point. Finally, the remaining cases where $|T / C|$ is not prime are when $p^{n}=3^{4}$. When $p^{n}=3^{4}$ and $G / C$ is not contained in the semilinear group, $G / C$ must be isomorphic to one of the Bucht groups $B_{0} \subseteq B_{1} \subseteq B_{2}$ which have orders 160, 320, 640 respectively. The stabilizer in $B_{2}$ of a nonzero vector of $V$ has order 8. Huppert and Blackburn give explicit generators $N_{1}, N_{2}, N_{3}, N_{4}, G, F$ for this group in XII Example 7.4 of [3]. The element $G^{3} N_{1}$ is an example of an element of order 8 which stabilizes a nontrivial point. The nontrivial point stabilizers in $B_{0}$ and $B_{1}$ are subgroups of the
nontrivial point stabilizers in $B_{2}$ so it follows that in all these cases the nontrivial point stabilizers are cyclic.

Note that there are few enough exceptional cases that it is also easy to verify that $T / C$ is either cyclic or isomorphic to $S_{3}$ by using group theory software such as MAGMA. In either case, $M \neq C$ as $T / M$ is not cyclic and $|T / M|$ is a square so is not equal to 6 .

We are now ready to show that under certain conditions, $G / N$ must contain more than one minimal normal subgroup.

Lemma 7.2. Assume the notation of Set-up 4.3 and that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$, $d>(e-1)^{2}$, and $e>1$. If either $(d / e+1, e)=1$ or $d>e^{5}-e$, then $G / N$ must contain another minimal normal subgroup distinct from $M / N$.

Proof. All the conditions necessary to apply both Lemma 6.2 and Lemma 7.1 as satisfied. Thus we have that $|G: T|=d / e,|T: M|=e^{2},|M: N|=d / e+1$, and $G$ is transitive on the nonidentity elements of $M / N$. Also $C=\mathbf{C}_{G}(M / N)$ has $M<C \subseteq T$ and we can thus take $K / M$ to be a $G$-chief factor contained in $C / M$. Both $M / N$ and $K / M$ are solvable chief factors so we can write $|M: N|=p^{a}$ and $|K: M|=q^{b}$ for some primes $p$ and $q$. Note that $M<K \subseteq C \subseteq T$ so $|K: M|=q^{b}$ divides $|T: M|=e^{2}$. We will consider four different cases and in each case we will either find another minimal normal subgroup or show that there is a contradiction.

Case 1: $q \neq p$
If $q \neq p$, then $K / N$ is the product of its Sylow $p$-subgroup $M / N$ and a Sylow $q$ subgroup $Q / N$. As $M / N$ is central in $K / N$, we will have $Q / N$ normal in $K / N$. It is a normal Sylow subgroup and is thus characteristic so $Q / N$ is normal in $G / N$ and another minimal normal subgroup exists in this case.

Case 2: $q=p$ and $K / N$ is not abelian
We are now assuming that $q=p$ so $|K: M|=p^{b}$ and $K / N$ is a $p$-group. Note that $p$ divides both $e$ and $d / e+1$ in this case so $(d / e+1, e) \neq 1$ and we must have $d>e^{5}-e$. For convenience, we will use the bar notation to denote groups $\bmod N$ so we will write $\bar{K}$ for $K / N$.

As $K / M$ and $M / N$ are $G$-chief factors, there are no characteristic subgroups of $\bar{K}$ properly between $\bar{N}$ and $\bar{M}$ or between $\bar{M}$ and $\bar{K}$. As $\bar{K} / \bar{M}$ is abelian but $\bar{K}$ is not, it follows that $\bar{K}^{\prime}=\bar{M}$. Also, $\bar{K}$ is contained in the centralizer in $\bar{G}$ of $\bar{M}$ so $\bar{M} \subseteq \mathbf{Z}(\bar{K})$ and again as $\bar{K}$ is not abelian, it follows that $\mathbf{Z}(\bar{K})=\bar{M}$. Hence $\bar{K}^{\prime}=\mathbf{Z}(\bar{K})=\bar{M}$. Also, note that some nontrivial element of $\bar{M}$ must be a commutator and the transitivity of $G$ on the nonidentity elements of $\bar{M}$ implies that elements of $\bar{M}$ are all commutators. Define a map of sets $\alpha: \bar{K} / \bar{K}^{\prime} \times \bar{K} / \bar{K}^{\prime} \rightarrow \bar{K}^{\prime}$ by $\alpha\left(\bar{K}^{\prime} x, \bar{K}^{\prime} y\right)=[x, y]$ where $[x, y]$ denotes the commutator of $x$ and $y$. It is easy to see that $[x, y]=[u x, v y]$ if $u$ and $v$ are central in $\bar{K}$, so as $\bar{K}^{\prime}=\mathbf{Z}(\bar{K})$ we see that $\alpha$ is well defined. The image of $\alpha$ is
$\bar{K}^{\prime}$ as all elements of $\bar{K}^{\prime}$ are commutators. Thus $\alpha$ is a surjective map between finite sets so $\left|\bar{K}^{\prime}\right| \leq\left|\bar{K} / \bar{K}^{\prime} \times \bar{K} / \bar{K}^{\prime}\right|=\left|\bar{K}: \bar{K}^{\prime}\right|^{2}$. Then $\left|\bar{K}^{\prime}\right|=|M: N|=d / e+1$ and $\left|\bar{K}: \bar{K}^{\prime}\right|^{2}=|K: M|^{2} \leq e^{4}$ so we get that $d / e+1 \leq e^{4}$. This is equivalent to $d \leq e^{5}-e$ which is a contradiction.

Case 3: $q=p$ and $K / N$ abelian but not elementary abelian
As in the previous case, we must have $d>e^{5}-e$ and we will continue to use the bar notation to denote groups $\bmod N$. Let $\varphi: \bar{K} \rightarrow \bar{K}$ be the $p$-th power map. As $\bar{K}$ is abelian, this is a group homomorphism. The kernel of $\varphi$ is exactly the elements of $\bar{K}$ of order $p$. This contains $\bar{M}$ but cannot be all of $\bar{K}$ and as $\operatorname{ker}(\varphi)$ is a characteristic subgroup of $\bar{K}$, it must equal $\bar{M}$. The image of $\varphi$ is the $p$-th powers of elements of $\bar{K}$. It is thus contained in $\bar{M}$ and is nontrivial and characteristic so it equals $\bar{M}$. Thus $\bar{M}$ is both the kernel and image of $\varphi$. It follows that $|K: M|=|M: N|$ so $d / e+1=p^{b} \leq e^{2}$. We thus have $d \leq e^{3}-e<e^{5}-e$ which is a contradiction.

Case 4: $q=p$ and $K / N$ elementary abelian
As in the previous cases, we must have $d>e^{5}-e$ and we will continue to use the bar notation to denote groups mod $N$. Note that $|G: K|=|G: T||T: M| /|K: M|=$ $(d / e) e^{2} / p^{b}$. Let $p^{c}$ be the largest power of $p$ dividing $|G: K|$. As $d / e+1=p^{a}$ we know that $p$ does not divide $d / e$ so $p^{c}$ divides $e^{2} / p^{b}$. Let $H / K$ be a Hall $p^{\prime}$-subgroup of $G / K$ so $|G: H|=p^{c}$ and $|H: K|=(d / e)\left(e^{2} / p^{b+c}\right)$. As $\bar{K}$ is abelian, $H / K$ acts on $\bar{K}$ by conjugation and this is a coprime action as $(|H / K|,|\bar{K}|)=1$. By Maschke's Theorem (1.9 of [5]), $\bar{M}$ has a complement $\bar{P}$ in $\bar{K}$ which is normal in $\bar{H}$. As $\bar{P}$ complements $\bar{M}$ in $\bar{K}$ we know that $\bar{M} \cap \bar{P}=1$ and $|\bar{P}|=|\bar{K}: \bar{M}|=p^{b}$. See the following diagram of indices.


If $\bar{P} \triangleleft \bar{G}$, then $P / N$ is a minimal normal subgroup of $G / N$ which is distinct from $M / N$ and we are done. Thus we can assume that $\bar{P}$ is not normal in $\bar{G}$ so there is some $G$-conjugate $\bar{R}$ of $\bar{P}$ which is not equal to $\bar{P}$. Let $r$ be an element of $\bar{R}$ which is not in $\bar{P}$. Then as $r \in \bar{K}=\overline{P M}$ we can write $r=x m$ for some $x \in \bar{P}$ and $m \in \bar{M}$ with
$m \neq 1$. Solving for $m$ we get that $m=x^{-1} r$ and thus some nonidentity element of $\bar{M}$ is contained in $\overline{P R}$. Note that all conjugates of $\bar{P}$ are contained in the abelian group $\bar{K}$ so the product of any two conjugates of $\bar{P}$ is a group. As $G$ is transitive on the nonidentity elements of $\bar{M}$ we see that every element of $\bar{M}$ is contained in a conjugate of $\overline{P R}$. It follows that every element of $\bar{M}$ is contained in some product of two conjugates of $\bar{P}$ so

$$
\bar{M} \subseteq \bigcup_{g, h \in G} \bar{P}^{g} \bar{P}^{h}
$$

The normalizer of $\bar{P}$ contains $\bar{H}$ which has index $p^{c} \leq e^{2} / p^{b}$ in $\bar{G}$ so there are at most $e^{2} / p^{b}$ conjugates of $\bar{P}$ in $\bar{G}$. Thus there are at most $\left(e^{2} / p^{b}\right)^{2}=e^{4} / p^{2 b}$ groups of the form $\bar{P}^{g} \bar{P}^{h}$. Each conjugate of $\bar{P}$ has order $p^{b}$ so each group of the form $\bar{P}^{g} \bar{P}^{h}$ has order at most $\left(p^{b}\right)^{2}=p^{2 b}$. This tells us that

$$
d / e+1=|M: N|=|\bar{M}| \leq\left|\bigcup_{g, h \in G} \bar{P}^{g} \bar{P}^{h}\right| \leq\left(e^{4} / p^{2 b}\right) p^{2 b}=e^{4}
$$

This would contradict that $d>e^{5}-e$ so $\bar{P}$ must be normal in $\bar{G}$.

## 8 Main Result

In Theorem 3.3 we showed that given two positive integers $d, e$ with $d / e+1$ a prime power, there is an example of a character triple $(G, N, \theta)$ and $\chi \in \operatorname{Irr}(G \mid \theta)$ such that $\chi$ has relative character degree $d$ and $|G: N|=d(d+e)$. For $d$ large enough, we will show that it is necessary that $d / e+1$ is a prime power. We will also show that there must be a sharply 2 -transitive group between $N$ and $G$.

Theorem 8.1. Let $(G, N, \theta)$ be a character triple with $G / N$ solvable and $\chi \in \operatorname{Irr}(G \mid \theta)$ with relative character degree $d$. Write $|G: N|=d(d+e)$ for some non-negative integer $e$. If $e \geq 1$ and $d>(e-1)^{2}$, then $e$ divides $d$ and $d / e+1$ is a prime power.

If in addition we have that $(d / e, e)=1,(d / e+1, e)=1$, or $d>e^{5}-e$, then there exist groups $X$ and $Y$ such $N \subseteq X \triangleleft Y \subseteq G$ and $Y / X$ is a sharply 2-transitive group of $\operatorname{order}(d / e)(d / e+1)$.

Proof. First assume that $e=1$. In Theorem 5.2 we showed that $G / N$ is a sharply 2 transitive group of order $d(d+1)$. Then $G / N$ has a normal subgroup $M / N$ of order $d+1$ by Lemma 5.1 and the transitive action of $G / N$ on the nonidentity elements of $M / N$ implies that $M / N$ is a $p$-group for some prime $p$. Thus $d / e+1=d+1$ is a prime power. The second result also holds for any $d$ with $X=N, Y=G$.

We will proceed by induction on $e$. Suppose now that $e>1$ and assume the notation of Set-up 4.3. We will consider the cases where $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$ and where $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ separately.

First assume that $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$. By Lemma 4.4, we have $e_{1}=e / q$ and $d_{1}=d / q$ for some integer $q>1$. As we have assumed $e \neq 0$, we see that $e_{1} \neq 0$ and $e_{1}<e$. We want to apply the inductive argument so we need to show that $d_{1}>\left(e_{1}-1\right)^{2}$. Note that $1<q$ so $(e-1)>(e-q)$ and as $q \leq e$ both sides of this inequality are non-negative so we can square both sides to get that $(e-1)^{2}>(e-q)^{2}$. This tells us that

$$
d_{1}=\frac{d}{q}>\frac{(e-1)^{2}}{q}>\frac{(e-1)^{2}}{q^{2}}>\frac{(e-q)^{2}}{q^{2}}=\left(\frac{e}{q}-1\right)^{2}=\left(e_{1}-1\right)^{2} .
$$

Thus we can apply induction to the character triple $(T, M, \varphi)$ to get that $e_{1}$ divides $d_{1}$ and $d_{1} / e_{1}+1$ is a prime power. Note that $d_{1} / e_{1}+1=(d / q)(e / q)+1=d / e+1$ so $e$ divides $d$ and $d / e+1$ is a prime power.

Furthermore, as $d_{1} / e_{1}=d / e$ and $e_{1}=e / q$ divides $e$ we see that if $(d / e, e)=1$, then $\left(d_{1} / e_{1}, e_{1}\right)=1$. Similarly if $(d / e+1, e)=1$, then $\left(d_{1} / e_{1}+1, e_{1}\right)=1$. If $d>e^{5}-e$, then we again use that $e_{1}=e / q$ and $d_{1}=d / q$ for some integer $q>1$ so

$$
d_{1}=\frac{d}{q}>\frac{e^{5}-e}{q}=\frac{e}{q}\left(e^{4}-1\right)=e_{1}\left(e^{4}-1\right)>e_{1}\left(e_{1}^{4}-1\right)=e_{1}^{5}-e_{1} .
$$

In all cases, we can therefore apply the inductive argument to get that there exists groups $X, Y$ such that $M \subseteq X \triangleleft Y \subseteq T$ with $|Y / X|=\left(d_{1} / e_{1}\right)\left(d_{1} / e_{1}+1\right)=(d / e)(d / e+1)$ and $Y / X$ is a sharply 2-transitive group. Clearly these groups also satisfy $N \subseteq X \triangleleft Y \subseteq G$ so they are also the desired groups for $(G, N, \theta)$.

Now assume that $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$. By Lemma 6.2 part ( $c$ ) we have that $|M / N|=d / e+1$ so in this case $e$ divides $d$ and $d / e+1$ is a prime power. By that same lemma, we also know that $|G: T|=d / e,|T: M|=e^{2}$ and $G$ is transitive on the nonidentity elements of $M / N$.

If either $(d / e+1, e)=1$ or $d>e^{5}-e$, then by Lemma 7.2 there is a second minimal normal subgroup of $G / N$ distinct from $M / N$. Call this other minimal normal subgroup $L / N$. By Lemma 6.1, we know that as $G$ is not transitive on $\operatorname{Irr}(M \mid \theta)$ we have that $M=N \mathbf{V}(\chi)$. If $G$ were not transitive on $\operatorname{Irr}(L \mid \theta)$, that same lemma would imply that $L=N \mathbf{V}(\chi)=M$ which would contradict that $M \neq L$. It follows that $G$ is transitive on $\operatorname{Irr}(L \mid \theta)$. The choice of $M / N$ was arbitrary so we work instead with $L / N$ and we have already shown that in this case we have the result by induction.

It remains to check the case where $(d / e, e)=1$. Let $\pi$ be the set of primes dividing $d / e$. As $(d / e, e)=1$ we know that none of these primes divides $e$. As $G / M$ is solvable, we can take $Y / M$ to be a Hall $\pi$-subgroup of $G / M$. Note that $|Y: M|=d / e$ as $|G: M|=(d / e) e^{2}$ and the primes in $\pi$ all divide $d / e$ but do not divide $e$. Consider the subgroup $Y / N$. This group has order $|Y: M||M: N|=(d / e)(d / e+1)$. Let $x$ be a nonidentity element of $M / N$ and note $x$ has $d / e$ conjugates under the action of $G$ since $G$ is transitive on the nonidentity elements of $M / N$ and $|M / N|=d / e+1$. The stabilizer of $x$ in $G / M$, call this $S / M$, has order $e^{2}$ as there are $d / e$ conjugates of $x$ and $|G: M|=(d / e) e^{2}$. Then $(|S: M|,|Y: M|)=\left(e^{2}, d / e\right)=1$ so $S / M \cap Y / M=1$ and the stabilizer of $x$ in $Y / M$ is 1 . Thus $x$ has $|Y / M|=d / e$ conjugates under the action of $Y$
so $Y$ is transitive on the nonidentity elements of $M / N$. By Lemma 5.1 the group $Y / N$ is a sharply 2-transitive group. Thus we get the desired result with $X=N$.

Note that if $e$ is a prime power, then as $(d / e, d / e+1)=1$ we must have that either $(d / e, e)=1$ or $(d / e+1, e)=1$.

Also, if $e$ is coprime to both $d / e$ and $d / e+1$, then we can show that the group $Y / X$ appears as a subgroup of $G / N$.

Corollary 8.2. Let $(G, N, \theta)$ be a character triple with $G / N$ solvable and $\chi \in \operatorname{Irr}(G \mid \theta)$ with relative character degree $d$. Write $|G: N|=d(d+e)$ for some non-negative integer $e$. Assume that $e \geq 1$ and $d>(e-1)^{2}$. Then $e$ divides $d$ and if $(e,(d / e)(d / e+1))=1$, then $G / N$ has a subgroup of order $(d / e)(d / e+1)$ which is a sharply 2-transitive group.

Proof. By Theorem 8.1 we know that $e$ divides $d$ and $d / e+1$ is a prime power. Also, as $(e,(d / e)(d / e+1))=1$ both $(e, d / e)=1$ and $(e, d / e+1)=1$ so there exists groups $X, Y$ such that $N \subseteq X \triangleleft Y \subseteq G$ and $Y / X$ is a sharply 2-transitive group of order $(d / e)(d / e+1)$. Note that $|X / N|$ must divide $e^{2}$ as

$$
(d / e)(d / e+1) e^{2}=|G: N|=|G: Y||Y: X||X: N|=|G: Y|(d / e)(d / e+1)|X: N| .
$$

The group $Y / N$ has normal subgroup $X / N$. The order of $X / N$ divides $e^{2}$ and the index of $X / N$ in $Y / N$ is $(d / e)(d / e+1)$ and hence $(|X / N|,|Y / N: X / N|)=1$ so by the SchurZassenhaus Theorem (Theorem 3.8 of [6]) we see that $X / N$ has a complement $Z / N$ in $Y / N$. As $Z / N$ complements $X / N$ is must be isomorphic to $(Y / N) /(X / N) \cong X / Y$ so $Z / N$ is the desired subgroup.

## $9 \quad G / N$ Nilpotent

Another interesting case to consider is what happens if we strengthen our assumption from $G / N$ solvable to $G / N$ nilpotent. For $e=0$, the example we built in Theorem 3.2 was a direct product of $p$-groups and thus nilpotent so the $e=0$ case does not change. When $e>0$, we showed in Theorem 8.1 that if $d>e^{5}-e$, then there exists groups $X$ and $Y$ such $N \subseteq X \triangleleft Y \subseteq G$ and $Y / X$ is a sharply 2-transitive group of order $(d / e)(d / e+1)$. With the exception of the group of order 2 , the sharply 2 -transitive groups are not nilpotent. It thus follows as a corollary to Theorem 8.1 that if $G / N$ is nilpotent and $e>0$, we either have $d=e=1$ or $d \leq e^{5}-e$. In particular, $d$ and thus $|G: N|$ is actually bounded in this case. The bound turns out to be much better than $e^{5}-e$ as we show in the next theorem.

Theorem 9.1. Let $(G, N, \theta)$ be a character triple with $G / N$ nilpotent and $\chi \in \operatorname{Irr}(G \mid \theta)$ with relative character degree $d$. Write $|G: N|=d(d+e)$ for some non-negative integer $e$. If $e>0$, then $d \leq e$ so $|G: N| \leq 2 e^{2}$.

Proof. We proceed by induction on $e$. If $e=1$, then by Theorem $5.2 G / N$ is a sharply 2-transitive group. By Lemma 5.1, we see that $G / N$ has a minimal normal subgroup $M / N$ of order $d+1$ and $G / N$ acts transitively on the $d$ nonidentity elements of $M / N$. In a nilpotent group, all minimal normal subgroups are central and hence this is only possible if $d=1=e$.

We next consider $e>1$. Assume the notation of Set-up 4.3 so that $M / N$ is a minimal normal subgroup of $G / N$ and $\varphi \in \operatorname{Irr}(M \mid \theta)$ is a constituent of $\chi_{M}$. Note that $M / N$ is central in $G / N$ so it has prime order. Looking at Lemma 4.1 we see that as $|M: N|$ is not a square, $\operatorname{Irr}(M \mid \theta)$ consists of $|M: N|$ distinct extensions of $\theta$.

Suppose that $G$ is not transitive on the members of $\operatorname{Irr}(M \mid \theta)$. By Lemma 4.6 there exists $\psi \in \operatorname{Irr}(M \mid \theta)$ which is $G$-invariant and $\lambda \in \operatorname{Irr}(M / N)$ such that $\varphi=\psi \lambda$. As $M / N$ is central in $G$, the irreducible characters of $M / N$ are fixed by $G$ so $\lambda$ is also $G$-invariant and hence $\varphi=\psi \lambda$ is $G$-invariant. As $\varphi$ is invariant in $G$ we have that $(G, M, \varphi)$ is a character triple with $\chi \in \operatorname{Irr}(G \mid \varphi)$. Also $\varphi(1)=\theta(1)$ so $d=\chi(1) / \theta(1)=\chi(1) / \varphi(1)$ and $\chi$ has relative character degree $d$ in $(G, M, \varphi)$. We can thus write $|G / M|=d\left(d+e_{1}\right)$ for some non-negative integer $e_{1}$. This gives us that

$$
d(d+e)=|G: N|=|G: M||M: N|=d\left(d+e_{1}\right) q
$$

and with some rearranging this becomes $e-e_{1} q=d(q-1)$. Both $d$ and $q-1$ are positive integers so $d(q-1)$ is a positive integer. Thus $e-e_{1} q$ is also a positive integer which is divisible by $d$ so $d \leq e-e_{1} q \leq e$.

Now suppose that $G$ is transitive on $\operatorname{Irr}(M \mid \theta)$. By Lemma 4.4, the character triple $(T, M, \varphi)$ has $d_{1}=d / q$ and $e_{1}=e / q$ where $q>1$ is an integer. As $e_{1}=e / q$ we see that $e_{1}>0$ and $e_{1}<e$ so we can apply induction to get that $d_{1} \leq e_{1}$ and as $d_{1}=d / q$ and $e_{1}=e / q$, this gives us $d \leq e$.

In particular, as $d \leq e$, we see that $|G: N|=d(d+e) \leq e(e+e)=2 e^{2}$.
This is in fact a sharp bound as we will show in the next lemma.
Lemma 9.2. For any positive integer e, there exists a character triple $(G, N, \theta)$ with $G / N$ nilpotent such that $e$ is a relative character degree of $(G, N, \theta)$ and $|G: N|=2 e^{2}$.

Proof. Let $G_{1}$ be the group of order 2. The triple $\left(G_{1}, 1,1\right)$ has relative character degree $d_{1}=1$ and if we write $2=\left|G_{1}\right|=d_{1}\left(d_{1}+e_{1}\right)=1\left(1+e_{1}\right)$, we get that $e_{1}=1$. As in Theorem 3.2, take $\left(G_{2}, N_{2}, \theta_{2}\right)$ to be such that $\theta_{2}$ is fully ramified in $G_{2}$ and with unique relative character degree $d_{2}=e$. If we write $\left|G_{2}: N_{2}\right|=d_{2}\left(d_{2}+e_{2}\right)$, then we have that $e_{2}=0$. Note that the example built in the proof of Theorem 3.2 was nilpotent.

We now take $(G, N, \theta)$ to be the direct product of the triples $\left(G_{1}, 1,1\right)$ and $\left(G_{2}, N_{2}, \theta_{2}\right)$. By Lemma 3.1, we have that $(G, N, \theta)$ has relative degree $d_{3}=d_{1} d_{2}=e$. Also, if we write $|G: N|=d_{3}\left(d_{3}+e_{3}\right)$, then $e_{3}=e_{1} d_{2}+e_{2} d_{1}+e_{1} e_{2}=e$. Thus $d_{3}=e_{3}=e$ and $|G: N|=e(e+e)=2 e^{2}$.

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