A Bound on the Order of a Group Having a Large Character Degree

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Abstract

Let d be the degree of an irreducible character of a finite group G. We can write $|G| = d(d+e)$ for some non-negative integer e. In this document, we prove that if $e > 1$ then $|G| < e^6 - e^4$. This improves an upper bound found by Isaacs of the form Be^6, where B is an unknown universal constant. We also describe conditions sufficient to sharpen this bound to $|G| \leq e^4 - e^3$. In addition, we remove the appeal to the classification of simple groups, which is used in the original paper by Isaacs.

1 Introduction

Let d be the degree of an irreducible character of a finite group G. We know that d divides $|G|$ and moreover that $d^2 \leq |G|$. We can therefore write $|G| = d(d+e)$ for some non-negative integer e. It is clear that if $e = 0$ then G must be the trivial group. Similarly the case where $e = 1$ has been fully classified; in fact, $e = 1$ if and only if either $|G| = 2$ or G is a 2-transitive Frobenius group. For a proof of this fact, see Proposition 2.1 of [4].

The case where $e = 1$ is in fact an anomaly as there is no upper bound on $|G|$ in this case. It was first proved by N. Snyder in [4] that for $e \geq 2$ there is an upper bound on $|G|$ in terms of e. The cases where $e = 2$ and $e = 3$ were also fully classified by Snyder, and Isaacs later proved in [3] that there is a polynomial bound for the order of G given by Be^6, where B is some universal constant. In order to obtain this bound, Isaacs needed to appeal to the classification of simple groups. In this document, we eliminate the unknown constant B to obtain a bound of $e^6 - e^4$, and we do so without an appeal to the classification. In addition, we see that it is possible in many cases to obtain the bound of $e^4 + e^3$ and we describe a condition sufficient to guarantee a bound of $e^4 - e^3$.

After a few definitions and technical theorems, our main result is the following. It appears as a corollary to a technical theorem.

Theorem A. *For $e > 1$ we have the following bounds on $|G|$ in terms of e;*

(a) If e is divisible by two distinct primes then $|G| < e^4 + e^3$.

(b) If e is a prime power then $|G| < e^6 - e^4$.

(c) If e is prime then $|G| < e^4 + e^3$.

We mention here that the largest group order we know of for a given e value is $|G| = e^4 - e^3$ where e is a prime power. This example is described in detail in Isaacs' paper [3]. In fact under certain conditions, we can show that this is the largest possible group order, as described in the following result.

Theorem B. *Suppose G has a non-trivial abelian normal subgroup.*

(a) If e is prime then $|G| \leq e^4 - e^3$.

(b) If e is divisible by two distinct primes then $|G| < e^4 - e^3$.

2 Assumptions and Basic Facts

We mention a few notational conventions which will help facilitate our discussion. Throughout this paper, we will assume $e \geq 1$. We fix $\chi \in \mathrm{Irr}(G)$ to have degree d, and we will consistently write $|G| = d(d+e)$. Notice that if $d \leq e$ then $|G| = d(d+e) \leq e(2e) = 2e^2$. We can therefore take $d > e$, where the situation becomes more interesting.

Lemma 2.1. *If $d > e$ then the character χ is rational valued, it is the unique character of largest degree in* $\mathrm{Irr}(G)$*, and* $\mathbf{Z}(\chi) = 1$*. In particular, χ is faithful.*

Proof. Since $d > e$, we have $|G| = d^2 + ed < 2d^2$. If $\psi \in \mathrm{Irr}(G)$ is different from χ and $\psi(1) \geq d$, then we would have $2d^2 \leq \chi(1)^2 + \psi(1)^2 \leq |G| < 2d^2$, a contradiction. Therefore d is the largest irreducible character degree and χ is unique of degree d. From this, it follows that χ is rational valued.

To see that χ is faithful, we use an argument presented both by Snyder in [4] and Isaacs in [3]. Let $x \in G$ with $x \neq 1$. By the second orthogonality relation, we have

$$0 = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)\psi(x) = d\chi(x) + \sum_{\psi \neq \chi} \psi(1)\psi(x)\,.$$

That is, $-d\chi(x) = \sum_{\psi \neq \chi} \psi(1)\psi(x)$, and taking absolute values yields

$$d|\chi(x)| \leq \sum_{\psi \neq \chi} \psi(1)|\psi(x)| \leq \sum_{\psi \neq \chi} \psi(1)^2 = |G| - \chi(1)^2 = ed\,.$$

Canceling d's from both sides tells us that $|\chi(x)| \leq e < d$ for all $x \neq 1$, so the assumption that $e < d$ allows us to conclude that χ has a trivial center and also that χ is faithful.

□

3 Character Domination and Main Result

Definition 3.1. Given $\alpha, \beta \in \mathrm{Irr}(G)$, we say that α **dominates** β (or β **is dominated by** α) if $\beta\alpha = \beta(1)\alpha$. (Note that this is equivalent to saying that α is the only irreducible constituent of $\beta\alpha$).

Throughout this paper, we refer to the vanishing-off subgroup of a character α, which we denote by $\mathbf{V}(\alpha)$. This is the subgroup of G generated by the elements $g \in G$ for which $\alpha(g) \neq 0$. Note that as α is constant on the conjugacy classes of G, this subgroup is necessarily normal in G.

Lemma 3.2. *Let $\alpha, \beta \in \mathrm{Irr}(G)$. Then α dominates β if and only if* $\mathbf{V}(\alpha) \subseteq \ker(\beta)$.

Proof. Suppose $\mathbf{V}(\alpha) \subseteq \ker(\beta)$. Then if $g \in \mathbf{V}(\alpha)$, we have $g \in \ker(\beta)$ so $(\beta\alpha)(g) = \beta(g)\alpha(g) = \beta(1)\alpha(g)$. If $g \notin \mathbf{V}(\alpha)$ then $\alpha(g) = 0$, so we have $(\beta\alpha)(g) = \beta(g)\alpha(g) = 0 = \beta(1)\alpha(g)$. In either case, $(\beta\alpha)(g) = \beta(1)\alpha(g)$ for all $g \in G$.

Conversely, suppose that $\beta\alpha = \beta(1)\alpha$. Then for any element $g \in G$ such that $\alpha(g) \neq 0$, we have $\beta(g)\alpha(g) = \beta(1)\alpha(g)$, so canceling $\alpha(g)$ gives $\beta(g) = \beta(1)$ and $g \in \ker(\beta)$. As $\mathbf{V}(\alpha)$ is generated by all $g \in G$ for which $\alpha(g) \neq 0$, this implies that $\mathbf{V}(\alpha) \subseteq \ker(\beta)$.

□

We now return to the situation where $|G| = d(d+e)$ and $\chi \in \mathrm{Irr}(G)$ has degree d. We continue to assume that $d > e$. For the rest of this section, we will write V for $\mathbf{V}(\chi) \subseteq G$.

Theorem 3.3. *Suppose there exists $\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$ such that χ does not dominate ψ. Then $d^3 \leq (de - |G : V|)^2$.*

Proof. Write $\{\psi_1, \ldots, \psi_n\}$ to denote $\mathrm{Irr}(G) \setminus \{\chi\}$. For $\psi_i \in \mathrm{Irr}(G) \setminus \{\chi\}$, write

$$\chi\psi_i = b_i\chi + \sum_{j=1}^{n} a_{ij}\psi_j \,,$$

where $a_{ij} = [\chi\psi_i, \psi_j]$ and $b_i = [\chi\psi_i, \chi]$. Evaluating $\chi\psi_i$ at the identity gives $d\psi_i(1) = db_i + \sum_{j=1}^{n} a_{ij}\psi_j(1)$, and then solving for b_i we get

$$b_i = \psi_i(1) - \frac{1}{d}\sum_{j=1}^{n} a_{ij}\psi_j(1) \,. \tag{1}$$

Next, consider the character χ^2. As χ is real-valued by Lemma 2.1, we know that $b_i = [\chi\psi_i, \chi] = [\psi_i, \chi^2]$. We can therefore write χ^2 as

$$\chi^2 = c\chi + \sum_{i=1}^{n} b_i\psi_i \,,$$

where $c = [\chi^2, \chi]$. Evaluating at the identity and plugging in the values for b_i from equation (1) gives

$$d^2 = cd + \sum_{i=1}^{n}\left(\psi_i(1) - \frac{1}{d}\sum_{j=1}^{n} a_{ij}\psi_j(1)\right)\psi_i(1) = cd + \sum_{i=1}^{n}\psi_i(1)^2 - \frac{1}{d}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_i(1)\psi_j(1) \,.$$

We can simplify the term $\sum_{i=1}^{n}\psi_i(1)^2$ by observing that it is the sum of the squares of all the irreducible characters other than χ. It follows that $\sum_{i=1}^{n}\psi_i(1)^2 = |G| - \chi(1)^2 = d(d+e) - d^2 = de$. Thus we have

$$d^2 = dc + de - \frac{1}{d}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_i(1)\psi_j(1) \,.$$

Some rearranging gives

$$d^2(c + e - d) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\psi_i(1)\psi_j(1) \,. \tag{2}$$

The right hand side of equation (2) must be a non-negative integer as all a_{ij} are non-negative integers. In fact, it is a positive integer if there exists some a_{ij} which is non-zero. Also, as c, d, and e are integers, equation (2) implies that d^2 divides the double sum $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\psi_i(1)\psi_j(1)$.

By assumption, there is some irreducible character other than χ which is not dominated by χ. Without loss, assume ψ_1 is not dominated by χ. Thus $\chi\psi_1$ must have a constituent other than χ so $a_{1j} \neq 0$ for some $1 \leq j \leq n$, and the right hand side of (2) is a positive integer divisible by d^2. In particular, we have

$$d^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\psi_i(1)\psi_j(1)\,. \tag{3}$$

We can order the ψ_i so that the first m of them are precisely the ones which are not dominated by χ, where $1 \leq m \leq n$. Then if ψ_i is dominated by χ, (i.e. if $i > m$) then $a_{ij} = 0$ for all $1 \leq j \leq n$. Also, as $a_{ij} = [\chi\psi_i, \psi_j] = [\psi_i, \chi\psi_j] = a_{ji}$, we have that $a_{ij} = 0$ if either i or j is greater than m. We can therefore simplify the right hand side of inequality (3) to get

$$d^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\psi_i(1)\psi_j(1)\,. \tag{4}$$

We now wish to find an upper bound for a_{ij} in terms of $\psi_i(1), \psi_j(1)$, and $\chi(1)$. By definition, $a_{ij} = [\chi\psi_i, \psi_j] = [\chi, \overline{\psi_i}\psi_j]$ so a_{ij} is the multiplicity of χ in $\overline{\psi_i}\psi_j$, and thus $a_{ij} \leq \dfrac{\overline{\psi_i}(1)\psi_j(1)}{\chi(1)} = \dfrac{\psi_i(1)\psi_j(1)}{d}$. Replacing a_{ij} with this upper bound in inequality (4) yields

$$d^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\psi_i(1)\psi_j(1) \leq \frac{1}{d} \sum_{i=1}^{m} \sum_{j=1}^{m} \psi_i(1)^2\psi_j(1)^2\,.$$

Multiplying both sides by d we get

$$d^3 \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \psi_i(1)^2\psi_j(1)^2 = \left(\sum_{i=1}^{m} \psi_i(1)^2\right)^2\,. \tag{5}$$

The characters ψ_i running from 1 to m are precisely the characters of $\mathrm{Irr}(G)\setminus\{\chi\}$ that are not dominated by χ, so by Lemma 3.2 they are exactly the characters in $\mathrm{Irr}(G)\setminus\{\chi\}$ which do not have V in their kernels. Thus

$$\sum_{i=1}^{m} \psi_i(1)^2 = |G| - \chi(1)^2 - |G:V| = d(d+e) - d^2 - |G:V| = de - |G:V|\,.$$

Replacing $\sum_{i=1}^{m} \psi_i(1)^2$ by $de - |G:V|$ in equation (5) gives the result

$$d^3 \leq (de - |G:V|)^2\,.$$

□

Corollary 3.4. *If $d \geq e^2$ then every element of* $\mathrm{Irr}(G) \setminus \{\chi\}$ *is dominated by* χ.

Proof. Suppose that at least one element of $\mathrm{Irr}(G) \setminus \{\chi\}$ is not dominated by χ. By Theorem 3.3, we have $d^3 \leq (de - |G:V|)^2$. In particular, as $|G:V| \geq 1$, we have $d^3 < (de)^2 = d^2e^2$ so $d < e^2$. As we are assuming that $d \geq e^2$, we must have that every element of $\mathrm{Irr}(G) \setminus \{\chi\}$ is dominated by χ. □

4 All Characters Dominated

In this section, we will discuss the case where χ dominates all the other irreducible characters of G. As in Section 3, we will write $V = \mathbf{V}(\chi)$.

Lemma 4.1. *If $\chi \in \mathrm{Irr}(G)$ dominates all the elements of $\mathrm{Irr}(G) \setminus \{\chi\}$ and $d > e$ then V is the unique minimal normal subgroup of G and V has order $(d/e) + 1$. Also, G acts transitively on the non-principal irreducible characters of V and V is an elementary abelian p-group for some prime p.*

Proof. First note that $V > 1$ since if $V = 1$, then χ is 0 on all non-identity elements of G, making $[\chi, 1_G] = \dfrac{1}{|G|}\chi(1) \neq 0$ and contradicting that $\chi \neq 1_G$. Hence V is a non-trivial normal subgroup of G. Also, by Lemma 3.2 we know V is contained in the kernel of every irreducible character that is dominated by χ. Thus V is contained in the kernel of every irreducible character except χ, which has trivial kernel by Lemma 2.1. Every non-trivial normal subgroup N is an intersection of non-trivial kernels of irreducible characters, which all contain V. This means that V is contained in every non-trivial normal subgroup of G and thus must be the unique minimal normal subgroup of G. Also, $|G:V|$ is the sum of the squares of the irreducible characters of G with V in their kernels. Thus $|G:V| = \sum_{\psi \neq \chi} \psi(1)^2 = |G| - \chi(1)^2 = d(d+e) - d^2 = de$. Rearranging terms, we find

$$|V| = \frac{|G|}{de} = \frac{d(d+e)}{de} = \frac{d}{e} + 1\,,$$

as wanted.

To show that V is abelian, we will show that all irreducible characters of V have degree 1. Let $\theta \in \mathrm{Irr}(V)$ with $\theta \neq 1_V$. If $\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$ then by assumption $V \subseteq \ker(\psi)$ so $0 = [\psi_V, \theta] = [\psi, \theta^G]$. Thus the only possible irreducible constituent of θ^G is χ, so $0 \neq [\theta^G, \chi] = [\theta, \chi_V]$. Hence every non-principal irreducible character of V is a constituent of χ_V. The constituents of χ_V form an orbit under the natural action of G on $\mathrm{Irr}(V)$ so G acts transitively on $\mathrm{Irr}(V) \setminus \{1_V\}$. Also, the irreducible constituents of χ_V must all have the same degree, say f. As every non-principal irreducible character of V has degree f, we can write $|V| = 1 + nf^2$ where $n = |\mathrm{Irr}(V) \setminus \{1_V\}|$. But f must divide $|V|$ and yet

$|V| \equiv 1 \pmod{f}$ so $f = 1$. Thus every character of V has degree 1 and V is abelian. In particular, since V is an abelian minimal normal subgroup of G, we see that V must be an elementary abelian p-group for some prime p. $\square$

To complete our analysis of this situation, we appeal to a theorem of Gagola. Although Gagola's original proof (see Theorem 6.2 in [1]) appeals to the classification of simple groups, a proof given by Isaacs in [3] does not require the classification. We state, but do not prove, the theorem here.

Theorem 4.2 (Gagola)**.** *Let $N \lhd G$, and assume that the natural action of G on the non-principal irreducible characters of N is transitive. Let θ be a non-principal character of N, and assume that θ is fully ramified in its stabilizer T in G. Then N is a p-group for some prime p, and $T \in \mathrm{Syl}_p(G)$. Also, if $T > N$, then $\mathbf{O}_p(G) = \mathbf{C}_G(N) > N$, and $|T : N| > |N|$.*

Theorem 4.3. *Assume $\chi \in \mathrm{Irr}(G)$ dominates all the characters in $\mathrm{Irr}(G) \setminus \{\chi\}$ and that $d > e > 1$. Then e and $(d/e)+1$ must be powers of the same prime p, and $(d/e)+1 < e^2$.*

Proof. As usual, let $V = \mathbf{V}(\chi)$. By Lemma 4.1, we know that V is the unique minimal normal subgroup of G. Furthermore, V is abelian and has order $(d/e) + 1$. Also, the natural action of G on the non-principal irreducible characters of V is transitive. Let $\lambda \in \mathrm{Irr}(V)$ be non-principal. Let T be the stabilizer of λ in G. As V is contained in the kernel of the action, we have a transitive action of G/V on $\mathrm{Irr}(V) \setminus \{1_V\}$ and the stabilizer of λ is T/V. Since $|\mathrm{Irr}(V) \setminus \{1_V\}| = |V| - 1 = (d/e)$, the Orbit-Stabilizer Theorem tells us that

$$|T/V| = \frac{|G/V|}{d/e} = \frac{de}{d/e} = e^2 .$$

For all $\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$ we have $V \subseteq \ker(\psi)$, so λ is not a constituent of ψ_V as λ is non-principal. It follows that χ is the only irreducible constituent of λ^G. By Clifford's Theorem, λ^T has only one irreducible constituent and we have that λ is fully ramified in T. The conditions of Gagola's Theorem are therefore satisfied and we conclude that $T \in \mathrm{Syl}_p(G)$ for some prime p. As $|T| = |T/V||V| = e^2((d/e) + 1)$, we know that e and $(d/e) + 1$ are powers of p. Finally, since $|T : V| = e^2 > 1$, we have $T > V$ and Gagola's Theorem gives us that $|V| < |T : V|$, so $(d/e) + 1 < e^2$. $\square$

We now state our first main result, which was stated previously in a similar form as Theorem A.

Corollary 4.4. *For $e > 1$ we have the following bounds on d and $|G|$ in terms of e.*

(a) If e is not a prime power then $d < e^2$ and $|G| < e^4 + e^3$.

(b) If e is a prime power then $d < e^3 - e$ and $|G| < e^6 - e^4$.

(c) If e is prime then $d < e^2$ and $|G| < e^4 + e^3$.

Proof. Suppose $d \geq e^2$. Corollary 3.4 tells us that every irreducible character other than χ is dominated by χ. As $e > 1$, we have $d \geq e^2 > e$ so the hypotheses of Theorem 4.3 are satisfied. One conclusion of this theorem is that e must be a prime power. If e is not a prime power, then this is a contradiction, so we must have $d < e^2$. This gives us the bound for d in part (a).

We can now assume that e is a prime power. Another conclusion of Theorem 4.3 is that $(d/e) + 1 < e^2$ and both $(d/e) + 1$ and e are powers of the same prime. Solving the inequality for d yields $d < e^3 - e$ which is the bound in part (b). In the case where e is prime, we have that $(d/e) + 1$ is a power of e so the inequality $(d/e) + 1 < e^2$ can be sharpened to $(d/e) + 1 \leq e$. Solving for d we get $d \leq e(e-1)$. This contradicts our original assumption that $d \geq e^2$. Hence if e is prime we must have $d < e^2$ which is the bound in part (c).

The bounds on the order of G follow immediately by plugging in the bounds obtained for d into $|G| = d(d + e)$. □

We can actually slightly improve on the bound in part (b) of Corollary 4.4. As $(d/e) + 1 < e^2$ and both sides of this inequality are integers, we can sharpen this to $(d/e) + 1 \leq e^2 - 1$. Solving for d gives $d \leq e^3 - 2e$ and the resulting bound on $|G|$ is $|G| = d(d+e) \leq e^6 - 3e^4 + 2e^2$. This bound, although less elegant than the bound given in Corollary 4.4, may be more useful when dealing with specific examples.

5 The Solvable Case

In this section, we concentrate on groups G which have a non-trivial abelian normal subgroup. In particular, we will see that this assumption allows us to obtain a better bound on d in terms of e. As in the previous section, we write V for $\mathbf{V}(\chi)$.

Lemma 5.1. *If $d \geq e(e-1)$ then every non-trivial normal subgroup of G has order at least e.*

Proof. If $e = 2$ then the statement of the lemma certainly holds, so we may assume $e > 2$. If $1 < N < G$ and $N \lhd G$ then we know that $e < e(e-1) \leq d$ and $|G : N| \leq ed$ since $|G| = d^2 + ed$ and χ is faithful by Lemma 2.1. Hence $|G| = |G : N||N| \leq ed|N|$. Since $|G| = d(d + e)$, we see that $d(d + e) \leq ed|N|$ and thus $\frac{d}{e} + 1 \leq |N|$. As $d \geq e(e-1)$, we see that $|N| = \frac{d}{e} + 1 \geq \frac{e(e-1)}{e} + 1$, and hence $|N| \geq e$. □

Theorem 5.2. *Suppose G has a non-trivial abelian normal subgroup N, and assume that $e \geq 2$ and $d \geq e(e-1)$. Then $|G : V| \geq d$ and χ dominates all irreducible characters of G other than χ.*

Proof. Let $|N| = q$ and write $\chi_N = \frac{d}{t}\sum_{i=1}^{t}\lambda_i$, where the λ_i are G-conjugate. Since t is an orbit size on $\mathrm{Irr}(N)$, we know that $t \leq q$. In fact, we must have $t < q$ since 1_N is in its own orbit under the action of G. Setting $\lambda = \lambda_1$, we write T for the stabilizer of λ in G and $\eta \in \mathrm{Irr}(T)$ for the Clifford correspondent of χ and λ. Since $\eta^G = \chi$, we have $|G : T|\eta(1) = d$ and therefore $\eta(1) = d/t$. As $\eta(1)$ is a character degree, we conclude that t must divide d. By Ito's theorem applied to T, we know $\eta(1)$ divides $|T : N|$. Moreover, as $\eta_N = \eta(1)\lambda$, we have $|T : N| = \lambda^G(1) \geq \eta(1)^2$. We can therefore write $|T : N| = \eta(1)(\eta(1) + e_1)$ where e_1 is an integer and $e_1 \geq 0$.

We wish to determine e_1. Since $d(d + e) = |G|$ and $|G| = |G : T||T : N||N|$, we have that $d(d+e) = t\left(\frac{d}{t}\right)\left(\frac{d}{t} + e_1\right)q$. Simplifying this expression yields $d+e = \left(\frac{d}{t} + e_1\right)q$, or equivalently,

$$d\left(1 - \frac{q}{t}\right) + e = qe_1\,. \tag{6}$$

Since $t < q$, the quantity $(q/t) > 1$ and thus $(1 - (q/t)) < 0$. Hence $d\,(1 - (q/t)) + e < e$, and substituting this in to equation (6) gives us that $qe_1 < e$. As $q = |N|$ we know that q is not zero and by Lemma 5.1 we know that $q \geq e$. Hence $e_1 < e/q < 1$. We conclude that $e_1 = 0$, giving us that $|T : N| = (d/t)^2$ and we are in the fully ramified situation. That is, $\lambda^T = \dfrac{d}{t}\eta$, and as $\lambda^G = (\lambda^T)^G$, we have $\lambda^G = \left(\dfrac{d}{t}\eta\right)^G = \dfrac{d}{t}\chi$. Since $N \lhd G$, we conclude that $(d/t)\chi(g) = \lambda^G(g) = 0$ if $g \notin N$, and we see that χ vanishes off of N. This gives us that $V \subseteq N$ so V is an abelian normal subgroup of G.

As V is abelian, we know by Ito's theorem that d divides $|G : V|$. In particular, $|G : V| \geq d$. We claim that this forces all $\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$ to be dominated by χ. Otherwise, applying the inequality of Theorem 3.3 yields

$$d^3 \leq (de - |G : V|)^2 \leq (de - d)^2 = d^2(e - 1)^2\,.$$

If we cancel d^2 from both sides of this inequality, we then have $d \leq (e-1)^2$, which would contradict our hypothesis that $d \geq e(e - 1)$ as $(e - 1)^2 < e(e - 1)$ when $e \geq 2$. □

We may now derive our second main result, Theorem B, which we state again here for the convenience of the reader.

Corollary 5.3. *Suppose G has a non-trivial abelian normal subgroup N.*

(a) If e is prime then $d \leq e(e - 1)$ and $|G| \leq e^4 - e^3$.

(b) If e is divisible by two distinct primes then $d < e(e - 1)$ and $|G| < e^4 - e^3$.

Proof. If $d \geq e(e - 1)$ then Theorem 5.2 ensures that χ dominates all characters of $\mathrm{Irr}(G) \setminus \{\chi\}$. By Theorem 4.3, we know that e must be a prime power, so this gives us a contradiction if e is divisible by two distinct primes. Hence if we are in the situation of (b) then we must have $d < e(e - 1)$, implying that $|G| < e^4 - e^3$.

We may therefore assume that e is a prime power. Specifically, if e is prime then Corollary 4.4 gives us that $d < e^2$. Therefore if $d > e(e - 1)$ we have $e(e - 1) < d < e^2$. As Theorem 4.3 ensures that e divides d, we have a contradiction in this case. Thus we must have $d \leq e(e - 1)$ if e is prime, and $|G| \leq e^4 - e^3$. □

Note that as we do not have the bound $d < e^2$ when e is a prime power that is not prime, the methods of this section do not help to improve the bound of $d < e^3 - e$ obtained in Corollary 4.4. Recall that for $e = p^a$ with $a > 0$ and p prime we know of a class of examples of groups which have an irreducible character of degree $d = e(e-1)$. This example is discussed in detain in Isaacs' paper (see the introduction of [3]), and shows that the bound $d \leq e(e-1)$ of Corollary 5.3(a) is sharp. When e is divisible by two or more distinct primes, we know of no such family of examples, so it would be interesting to obtain a better feel for a bound on d in this case.

Note that Corollary 5.3 certainly applies when the group G is solvable, so we have the bound $|G| \leq e^4 - e^3$ for G solvable and e either prime or divisible by two distinct primes. Moreover, if we assume that e is either prime or divisible by two distinct primes and that $d > e(e-1)$, then we can conclude that G has no non-trivial solvable normal subgroups, since any such normal subgroup would have a non-trivial characteristic abelian subgroup.

6 Small e values

In this section we show how our work can be applied in practice. With the results of this paper in hand, it is possible to determine all possible groups that occur having relatively small e values. Snyder classified all groups with $e = 2$ and $e = 3$ in [4], and with our results it is possible to fully classify groups with e values of $4, 5$, and 6. We can also classify all possible d values which can occur with $e = 7$.

In the cases of $e = 5$ and $e = 6$, we use Corollary 4.4 to see that any groups having these e values must have orders less than 750 and 1512, respectively. In any case, computer software such as MAGMA can perform an extensive search to count all such non-isomorphic groups with these e values.

When $e = 5$, we find a total of 10 non-isomorphic groups having $|G| = d(d+5)$. More specifically, we have one abelian group of order 6 with $d = 1$ and one group where $d = 2$, yielding a group of order 14. There are three non-isomorphic groups of order 24 having an irreducible character of degree 3 and we have two non-isomorphic groups of order 36 with a character of degree 4. Finally, we have three distinct groups of order 500 which have a character of degree 20, one of which is the example given by Isaacs in his paper. This fully classifies all groups with $|G| = d(d+5)$ where $d = \chi(1)$ for some $\chi \in \mathrm{Irr}(G)$.

When $e = 6$, we determine that there are 18 distinct groups of order $d(d+6)$. Also, we find here that the largest d can be is 6, and we are lacking in an example where d is somewhat large in comparison to e. A more specific breakdown of the examples is as follows. When $d = 1$ we have $|G| = 1(1+6)$ and G is the unique group of order 7. For $d = 2$ we find nine groups of order 16 having a character of degree 2. We have two groups of order 27 with a character of degree 3 and two groups of order 40 having an irreducible character of degree 4. There is a unique group of order 55 with a character of degree 5. When $d = 6$, we have three non-isomorphic groups of order 72, and this is the largest d value possible for $e = 6$.

The case of $e = 4$ is somewhat more complicated as our techniques are less effective for prime powers which are not prime. However, it is still feasible to finish this case by hand.

Theorem 6.1. *There are 17 non-isomorphic groups G having order $d(d+4)$ for d the degree of an irreducible character of G.*

Proof. When $e = 4$, Corollary 3.4 tells us that if $d \geq 16$ then χ dominates all characters in $\mathrm{Irr}(G) \setminus \{\chi\}$. Applying Lemma 4.1 then tells us that G has a unique minimal normal subgroup N with order $(d/4)+1$ and this subgroup satisfies the hypotheses of Gagola's Theorem. For $\lambda \in \mathrm{Irr}(N) \setminus \{1_N\}$, write T for the inertia subgroup of λ. By Gagola's Theorem, we know $|G : N| = 4d$ and $|G : T| = d/4$, making $|T : N| = 16$. Furthermore, we know that T is a Sylow p-subgroup of G and since 2 divides $|T|$ we must have $T \in \mathrm{Syl}_2(G)$. Finally, we know that $|N| < |T : N|$ so we must have $|N| \in \{2, 4, 8\}$. As we are assuming that $d \geq 16$, however, we see that $|N| = (d/4) + 1 \geq 4 + 1 = 5$. We may therefore assume that $|N| = 8$, so that $d = 28$ and $|G| = 896$, which is a case that can be handled by MAGMA. A search of the Small Groups database for MAGMA shows that no group of order 896 has a character of degree 28. We can therefore assume that $d < 16$ and a search of the same database gives us the count of 17 non-isomorphic groups having $e = 4$. We find that the possible d values for $e = 4$ are $d = 1, 2, 3, 4$, and 12. The number of non-isomorphic groups having these d values are $1, 2, 1, 7$, and 6, respectively. $\square$

In the case of $e = 7$, we must check all possible orders of groups having $d < 49$. This means we must check all such groups having order less than 2640. Again, we rely on software such as MAGMA to classify all groups up to order 2000 having this e value, and we use ad hoc methods to eliminate other possibilities. For $e = 7$ the computer can check for groups satisfying $|G| = d(d+7)$ up to and including $d = 41$. This search gives 10 non-isomorphic groups with d values less than 42. The d values that occur are $1, 2, 5, 6, 8$, and 9, and the number of non-isomorphic groups for these d values are $3, 3, 1, 1, 1$, and 1, respectively. When $d = 42$, we see that $42 = 7 \cdot 6 = e(e-1)$, so we know that there is at least one group of order 2058 possessing an irreducible character of degree 42.

We can use the methods of the previous section to show that there are no groups with $|G| = d(d+7)$ and $43 \leq d \leq 47$. Since $e = 7$ is prime, we can apply Corollary 5.3 to conclude that if $d \geq 43$ then G cannot have any non-trivial abelian normal subgroups. In particular, G can have no normal Sylow p-subgroups for any prime p dividing $|G|$ since any such normal subgroup would have a non-trivial center, which would then be normal in G. For $d \in \{43, 44, 45, 46, 47\}$, we find that groups with $|G| = d(d+7)$ have normal Sylow 43,17,13,53, and 47 subgroups, respectively, so none of these situations can occur. The final case, where $d = 48$, is slightly more complicated, so we include a more detailed argument.

Lemma 6.2. *There is no group G of order 2640 having an irreducible character of degree 48.*

Proof. Since $|G| = 2640 = 2^4 \cdot 3 \cdot 5 \cdot 11$, we know by the Sylow counting Theorem that G has either a normal Sylow 11-subgroup or G has 12 Sylow 11-subgroups. As previously noted, because 7 is prime and $|G| > 2050 = 7^4 - 7^3$, we conclude that G cannot have a normal Sylow 11-subgroup by Corollary 5.3. We may therefore assume that G has 12 Sylow-11 subgroups. Let $N = \mathbf{N}_G(P)$ for $P \in \mathrm{Syl}_{11}(G)$, so that $|G : N| = 12$ and $|N| = 220 = 2^2 \cdot 5 \cdot 11$. Consider the action of G on the 12 cosets of N. The kernel of this action is $K = \mathrm{core}_G(N)$. As $K \lhd G$, there cannot be any non-trivial abelian characteristic subgroups of K as these would be abelian normal subgroups of G. If 11 divides $|K|$ then $P \subseteq K$ would be a non-trivial abelian characteristic subgroup of K. We can therefore assume that 11 does not divide $|K|$ so that $|K| \in \{1, 2, 4, 5, 10, 20\}$. Any such K is necessarily solvable and thus if $K > 1$ we know that K contains a non-trivial abelian characteristic subgroup. We must therefore have $K = 1$ so that G injects into S_{12}. If we write $C = \mathbf{C}_G(P)$, we can apply the "N/C Theorem" to conclude that N/C is isomorphic to a subgroup of the automorphism group of P. As $|\mathrm{Aut}(P)| = 10$, we see that 2 must divide $|C|$. Hence there is an element of order 2 commuting with all of P and we see that G has an element of order 22. However, S_{12} does not have an element of order 22 and as G injects into S_{12}, we have reached a contradiction. $\square$

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