# DISTINGUISHING GRAPHS WITH ZETA FUNCTIONS AND GENERALIZED SPECTRA 

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#### Abstract

Conjecturally, almost all graphs are determined by their spectra. This problem has also been studied for variants such as the spectra of the Laplacian and signless Laplacian. Here we consider the problem of determining graphs with Ihara and Bartholdi zeta functions, which are also computable in polynomial time. These zeta functions are geometrically motivated, but can be viewed as certain generalizations of characteristic polynomials. After discussing some graph properties determined by zeta functions, we show that large classes of cospectral graphs can be distinguished with zeta functions and enumerate graphs distinguished by zeta functions on $\leq 11$ vertices. This leads us to conjecture that almost all graphs which are not determined by their spectrum are determined by zeta functions.

Along the way, we make some observations about the usual types of spectra and disprove a conjecture of Setyadi and Storm about Ihara zeta functions determining degree sequences.


## 1. Introduction

A fundamental problem in spectral graph theory is: when can we distinguish (unlabeled simple) graphs by their spectra? If the answer were always, as was once conjectured, that would mean we could solve the graph isomorphism problem in polynomial time. However, many pairs of non-isomorphic cospectral graphs have since been found, and various constructions of cospectral pairs are known. Nevertheless, Haemers conjectured that almost all graphs are determined by their spectra (DS), i.e., the fraction of graphs of order $n$ which are DS goes to 1 as $n \rightarrow \infty$. In fact, from the numerical data for $n \leq 11$ in Haemers-Spence [HS04], it appears that more graphs are determined by their Laplacian spectra ( $L$-DS), and even more by their signless Laplacian spectra ( $|L|-\mathrm{DS}$ ). We refer to surveys by van Dam and Haemers for more details [vDH03], [vDH09].

This question of which graphs are DS can be thought of geometrically. Knowing the adjacency spectrum of $G$ is equivalent to knowing what one might call the walk length spectrum of $G$-the set $\left\{\left(\ell, w_{G}(\ell)\right): \ell \geq 0\right\}$, where $w_{G}(\ell)$ is the number of closed walks of length $\ell$ on $G$ together with the order $n$ of $G$. Hence the above question can be stated: when do the order and walk length spectrum determine $G$ ?

From the point of view of Riemannian geometry and covering space theory, it is more natural to look at geodesics on $G$ rather than arbitrary walks. Roughly, geodesics are paths with no backtracking and they correspond to lines in the universal cover of $G$ (see Section 2). The geodesic length spectrum of $G$ is then the set $\left\{\left(\ell, a_{G}(\ell)\right): \ell \geq 0\right\}$ of numbers of (primitive) closed geodesics of length $\ell$ for all $\ell$. For graphs, the Ihara zeta function $Z_{G}(t)$ of $G$ encodes the geodesic length spectrum, and knowing one is equivalent to knowing the other.

Date: April 28, 2015.
2010 Mathematics Subject Classification. 05C50, 05C30.
Key words and phrases. Ihara zeta function, Bartholdi zeta function, cospectral graphs.
*Partially supported by a Simons Collaboration Grant.

This is entirely analogous to the Selberg zeta function encoding the (geodesic) length spectrum for Riemann surfaces. For compact Riemann surfaces, Huber's theorem says that knowing the Selberg zeta function, i.e., the length spectrum, is equivalent to knowing the spectrum of the Laplacian (see, e.g., [Bus10]). Similarly, for connected regular graphs, knowing the Ihara zeta function is equivalent to knowing the (adjacency or Laplacian) spectrum (see Section 2.1). However, this is not the case for irregular graphs (cf. Table 5.2).

Here, we suggest that the Ihara zeta function-which is computable in polynomial timeprovides a more effective way to differentiate (irregular) graphs than the usual spectra studied (Conjecture 5.1). One heuristic for why this should be the case is that geodesics capture much of the geometry of the graph better than arbitrary paths. Another is that the Ihara zeta function typically encodes more information-at least for md2 graphs (graphs with no vertices of degree $\leq 1$ ), the reciprocal of the Ihara zeta function is a polynomial whose degree is twice the number of edges, and thus typically has more coefficients than the characteristic polynomial.

The obvious drawback of the Ihara zeta function is that it cannot detect leaves or isolated nodes. For this reason (at least in part), most studies of the Ihara zeta function restrict to md2 graphs. Most studies also restrict to connected graphs, which is convenient to discuss covering space theory, but this is less crucial for the problem of distinguishing graphs. (As with characteristic polynomials, the zeta function of a disconnected graph is the product of the zeta functions of its components.) Instead of making such restrictions here, we consider four methods of addressing this defect to distinguish graphs (undefined notation explained subsequently):
(M1) Use the order and the Ihara zeta $Z_{G^{*}}(t)$ function of the cone $G^{*}$ of $G$. Equivalently use the order and the characteristic polynomial $\varphi_{T^{*}}(\lambda)=\operatorname{det}\left(\lambda I-T^{*}\right)$.
(M2) Use the order, size and the Ihara zeta functions $Z_{G}(t)$ and $Z_{\bar{G}}(t)$ of $G$ and its complement $\bar{G}$. Equivalently, use the order and the characteristic polynomials $\varphi_{T}(\lambda)=\operatorname{det}(\lambda I-T)$ and $\varphi_{\bar{T}}(\lambda)=\operatorname{det}(\lambda I-\bar{T})$.
(M3) Use the order and the Bartholdi zeta function $\mathcal{Z}_{G}(t, u)$ of $G$. Equivalently, use the generalized characteristic polynomial $\varphi_{A D}(\lambda, x)=\operatorname{det}(\lambda I-A+x D)$.
(M4) Use the order and the Bartholdi zeta functions $\mathcal{Z}_{G}(t, u)$ and $\mathcal{Z}_{\bar{G}}(t, u)$ of $G$ and $\bar{G}$. Equivalently, use the generalized characteristic polynomial $\varphi_{A D J}(\lambda, x, y)=\operatorname{det}(\lambda I-A+x D+$ $y J)$.

Here $I$ and $J$ denote the identity and all-ones matrices of the appropriate sizes, and $A$ and $D$ denote the adjacency and degree matrices for a fixed ordering of vertices on $G$. Further, $T$ (resp. $T^{*}, \bar{T}$ ) denotes Hashimoto's oriented edge matrix (see Section 2) of $G$ (resp. $G^{*}, \bar{G}$ )whose spectrum contains the same information as $Z_{G}$ together with the size of $G$ (cf. (2.3)). The two-variable Bartholdi zeta function $\mathcal{Z}_{G}(t, u)$ is a generalization of the one-variable Ihara function $Z_{G}(t): \mathbb{C} \rightarrow \mathbb{C}$ which counts "closed geodesics with $r$ backtracks"-see Section 3, in particular Equation (3.3) for the relation with $\varphi_{A D}$.

Note that knowing $\varphi_{A D}$ implies knowing the spectra of $A$, the Laplacian $L$, and the signless Laplacian $|L|$. Similarly knowing $\varphi_{A D J}$ implies knowing all of these spectra for both $G$ and $\bar{G}$.

We will explain the precise motivation behind these choices as we consider each in turn in the body of the paper. The basic ideas are that the Ihara zeta function cannot detect "dangling edges," i.e., edges to a degree 1 node, but (i) taking cones or complements turns degree 1 nodes into higher degree nodes and (ii) the Bartholdi zeta function does detect dangling edges. In analogy with the DS terminology, we say that a graph $G$ is $\mathrm{DZ}^{*}$ (resp. $\mathrm{DZ} \bar{Z}, \mathrm{D} \mathcal{Z}, \mathrm{D} \mathcal{Z} \overline{\mathcal{Z}}$ ) if it is uniquely determined by method (M1) (resp. (M2), (M3), (M4)).

We will see in Section 3.2 that (M4) is provably stronger than each of (M1), (M2) and (M3), i.e., being $\mathrm{DZ}^{*}$ (resp. $\mathrm{DZ} \bar{Z}, \mathrm{D} \mathcal{Z}$ ) implies $\mathrm{D} \mathcal{Z} \overline{\mathcal{Z}}$. However there are no obvious implications among (M1), (M2) and (M3) (but there are some indications that (M2) may be stronger than
(M1)—see the end of Section 5). Nevertheless, a surprising outcome of our calculations in Table 5.1 is that for graphs on $n \leq 11$ vertices, methods (M1), (M2) and (M4) are always equivalent (whereas (M3) has slightly less discriminating power). This suggests one gains little by using the Bartholdi zeta function, and one is in practice justified in just considering the Ihara zeta function, which is both conceptually and computationally simpler. However, all of these methods run in polynomial time.

Now we outline the contents of the paper.
In Section 2, we first recall basic facts about the Ihara zeta function, and discuss some graph properties it determines. Then we consider some basic properties determined by methods (M1) and (M2). In particular, these zeta invariants force strong restrictions on the degree sequence of a graph. Setyadi-Storm [SS13], in their enumeration of pairs of connected md2 graphs with the same Ihara zeta function on $n \leq 11$ vertices, found that the Ihara zeta function of a connected md2 graph determines the degree sequence for $n \leq 11$, and conjectured this holds for all $n$, but we give a counterexample to this conjecture on 12 vertices (Example 2.2). Nevertheless, we show that knowing the zeta functions of sufficiently many cones of $G$ algorithmically determines the degree sequence (Lemma 2.3).

In Section 3, we recall basic facts about the Bartholdi zeta function and discuss the relative strengths of methods (M1)-(M4). To show that (M4) is stronger than (M1), we show that two graphs have the same spectra with respect to $A+x D$ and $\bar{A}+x \bar{D}$ for fixed $x$, if and only if the same is true of their joins with another graph (Theorem 3.2). In particular, two graphs have the same $A$ - and $\bar{A}$ - (or $|L|$ - and $|\overline{L \mid}|$-) spectra if and only if the same is true for their cones (Corollary 3.3; cf. Table 5.4 for related data).

In Section 4, we show that methods (M1)-(M4) distinguish a large class of cospectral and Laplacian cospectral pairs coming from well known constructions: GM switching, coalescence and join. On the other hand, these methods will not distinguish graphs arising from the more restrictive $\mathrm{GM}^{*}$ switching. Section 4.4 presents a new construction of pairs of graphs which cannot be distinguished by these methods. This construction is interesting because it constructs graphs $G_{1}, G_{2}$ such that the generalized adjacency matrices $A_{1}+x D_{1}$ and $A_{2}+x D_{2}$ are miraculously conjugate for all $x$ but not "uniformly conjugate" (as happens in $\mathrm{GM}^{*}$ switching) i.e., there is no invertible matrix $P$ such that $P\left(A_{2}+x D_{2}\right) P^{-1}=A_{1}+x D_{1}$ for all $x$.

In Section 5, we enumerate all graphs on $\leq 11$ vertices which are not $\mathrm{DZ}^{*}$ (resp. $\mathrm{DZ} \overline{\mathrm{Z}}, \mathrm{D} \mathcal{Z}$, $\mathrm{D} \mathcal{Z} \overline{\mathcal{Z}}$ ) in Table 5.1. We state Conjecture 5.1, which asserts almost all graphs are $\mathrm{DZ}^{*}$ (resp. $\mathrm{DZ} \bar{Z}$, $\mathrm{D} \mathcal{Z}, \mathrm{D} \mathcal{Z} \overline{\mathcal{Z}}$ ), and further that almost all non-DS graphs are $\mathrm{DZ}^{*}$ (resp. DZZ $, \mathrm{D} \mathcal{Z}, \mathrm{D} \mathcal{Z} \overline{\mathcal{Z}}$ ). Table 5.2 indicates that there is no essential difference in just using the Ihara zeta function $Z_{G}$ compared to methods (M1)-(M4) when restricting to md2 graphs. This suggests Conjecture 5.2, which is the analogue of Conjecture 5.1 for determining md2 graphs $G$ using only $Z_{G}$. We also compare the effectiveness of combining different kinds of spectra for $n \leq 11$ in Table 5.3. In particular, this suggests that using two of the usual spectra, such as $A$ and $L$ or $A$ and $|L|$, is much more effective at distinguishing graphs than any single one.

Our calculations were done using Sage [Sage], including nauty [nauty], and standard Unix tools. In practice, we found that to check if two graphs have the same zeta function, it almost always sufficed to compute the two numbers $\operatorname{det}|L|$ and $\operatorname{det}(4 D+2 A-3 I)$, which are essentially (the residue of) $Z_{G}(1)$ and $Z_{G}(-2)$. For convenience of the interested reader, when mentioning particular examples of graphs, we include (non-canonical) graph 6 strings to specify the graphs, which can be used to easily reconstruct the graphs in Sage.

As a final remark, we note that there are more general notions of zeta functions of graphs, such as path and edge zeta functions (see [ST96] or [Ter11]). We do not consider these here.

We thank C. Storm for discussions about [SS13], as well as the referee for helpful comments.

## 2. The Ihara zeta function

We begin by defining closed geodesics and the Ihara zeta function. This notion of a geodesic on a graph corresponds to a bi-infinite simple path in the universal cover, but we do not explain this here-see, e.g., [ST96] or [Ter11]. Ihara zeta functions are a special case of more general zeta functions of multigraphs considered by Hashimoto [Has89], generalizing the zeta functions originally defined by Ihara [Tha66].

Fix a finite (simple) graph $G=(V, E)$. Let $n=|V|$ and $m=|E|$. We will denote walks by sequences of adjacent oriented (or directed) edges $e_{i}$. For an oriented edge $e_{i}=(u, v)$, let $e_{i}^{-1}=(v, u)$ be the edge with reversed orientation. Suppose $\gamma=\left(e_{1}, e_{2}, \ldots, e_{\ell}\right)$ denotes a closed walk of length $\ell$ in $G$. We say $\gamma$ is a closed geodesic if $e_{i} \neq e_{i+1}^{-1}$ for $1 \leq i<\ell$ and $e_{1} \neq e_{\ell}^{-1}$. The former condition is often expressed saying $\gamma$ has no backtracking and the latter that $\gamma$ has no tails. Write $k \gamma$ for the concatenation of $k$ copies of $\gamma$. We say $\gamma$ is primitive if $\gamma$ is not of the form $k \delta$ for a closed geodesic $\delta$ and some $k \geq 2$.

Let $\sigma(\gamma)=\left(e_{2}, \ldots, e_{\ell}, e_{1}\right)$. Then $\sigma(\gamma)$ is also a closed geodesic of length $\ell$, and it is primitive if $\gamma$ is. Thus the cyclic group $\langle\sigma\rangle$ of order $\ell$ generated by $\sigma$ acts on all (primitive) closed geodesics of length $\ell$. The $\langle\sigma\rangle$-orbits thus partition the set of (primitive) closed geodesics of length $\ell$ into equivalence classes. Let $a(\ell)=a_{G}(\ell)$ denote the number of primitive closed geodesics of length $\ell$ up to equivalence.

Note that, as $G$ is simple, there are no closed geodesics of length $<3$. The closed geodesics of length 3,4 or 5 are just the cycles of length 3,4 or 5 . The closed geodesics of length 6 are just the cycles of length 6 together with the concatenations of 2 simple cycles of length 3 starting at a fixed base point. If there are two distinct cycles based at a vertex $v_{0}$, going around the first cycle $k$ times and going around the second cycle once or more is a primitive closed geodesic. Hence, for connected md2 graphs, the lengths of primitive closed geodesics are unbounded unless $G$ is a circuit.

The (Ihara) zeta function of $G$ is

$$
\begin{equation*}
Z_{G}(t)=\prod_{\gamma}\left(1-t^{\ell(\gamma)}\right)^{-1}=\prod_{\ell>2}\left(1-t^{\ell}\right)^{-a(\ell)}=\exp \left(\sum_{\ell>2} \sum_{k \geq 1} a(\ell) \frac{t^{\ell k}}{k}\right) \tag{2.1}
\end{equation*}
$$

where, in the first product, $\gamma$ runs over a set of representatives for the equivalences classes of primitive closed geodesics, and $\ell(\gamma)$ denotes the length of $\gamma$. Since no closed geodesics will involve degree 0 or degree 1 nodes, $Z_{G}(t)=Z_{G^{\dagger}}(t)$, where $G^{\dagger}$ is the "pruned graph" obtained by successively deleting degree 0 and degree 1 nodes until one is either left with an md2 graph or the null graph (the graph on 0 vertices).

Note if $G$ is a disjoint union of two subgraphs, $G=G_{1} \sqcup G_{2}$, then $a_{G}(\ell)=a_{G_{1}}(\ell)+a_{G_{2}}(\ell)$. Hence the zeta function of a graph is the product of the zeta functions of its connected components.

The zeta function is a priori an infinite product, but turns out to be a rational function and thus is meromorphic on $\mathbb{C}$. Namely, Hashimoto [Has89], [Has92] and Bass [Bas92] gave two determinant formulas for $Z_{G}$, which have been subsequently retreated many times (e.g., [ST96]). The Bass determinant formula [Bas92] is

$$
\begin{equation*}
Z_{G}(t)=\left(1-t^{2}\right)^{n-m} \operatorname{det}\left(I-t A+t^{2}(D-I)\right)^{-1} \tag{2.2}
\end{equation*}
$$

Note the right hand side is invariant under adding nodes of degree 0 or 1 to $G$.
Let $\left\{e_{1}, \ldots, e_{2 m}\right\}$ denote the set of oriented edges of $G$. The oriented edge matrix (with respect to this ordering of oriented edges) $T$ is the $2 m \times 2 m$ matrix whose $(i, j)$-entry is 1 if $e_{i}=(u, v)$, $e_{j}=(v, w)$ and $u \neq w$; or 0 otherwise. The Hashimoto determinant formula [Has89], [Has92]
(cf. [ST96, Thm 3]) is

$$
\begin{equation*}
Z_{G}(t)=\operatorname{det}(I-t T)^{-1}=\left(t^{2 m} \varphi_{T}\left(t^{-1}\right)\right)^{-1} \tag{2.3}
\end{equation*}
$$

where $\varphi_{T}(\lambda)=\operatorname{det}(\lambda I-T)$ is the characteristic polynomial of $T$. Again, one can check that $\operatorname{det}(I-t T)$ is invariant under adding degree 0 or 1 nodes to $G$.

Since $G^{\dagger} \neq G$ in general, $Z_{G}$ does not determine $m$ (or $n$ ). This means knowing $Z_{G}$ is not exactly the same as knowing the spectrum of $T$, but it almost is: $\varphi_{T}$ determines both $Z_{G}$ and $m$, and conversely. Indeed, the degree of $\varphi_{T}$ is $2 m$, so one can recover $Z_{G}$ from $\varphi_{T}$. The converse is obvious. This observation relates to method (M2).
2.1. Properties determined by the Ihara zeta function. Here we summarize some elementary graph properties determined by the Ihara zeta function. This question has previously been considered mainly for md2 and connected md2 multigraphs (e.g., [Cza05] and [Coo09]).

First, examining the coefficients of $t^{k}$ in the logarithm of (2.1) shows $Z_{G}$ determines $a(\ell)$ for all $\ell$, hence knowing $Z_{G}$ is the same as knowing the primitive geodesic length spectrum. Note that the number of non-primitive geodesics of a given length $l$ is the sum of the numbers of primitive geodesics of proper divisors of $l$. Hence the primitive geodesic length spectrum determines the full geodesic length spectrum and vice versa. Consequently $Z_{G}$ determines the number of cycles of length 3 , length 4 and length 5 in $G$. It is not true that $Z_{G}$ determines the number of cycles of length 6 .
Example 2.1. The pair of graphs HheadXZ and Hhf@eS | pictured below, each with 9 vertices and 18 edges, have the same zeta functions but a different number of cycles of length 6 (46 and 50).


We remark $Z_{G}(t)=1$ if and only if $G$ has no cycles, so $Z_{G}$ can determine whether a graph is a forest or not. The girth $g$ of $G$ will be the smallest $\ell$ such that $a(\ell) \neq 0$. So $Z_{G}$ determines $g$ and, if $G$ is connected, Scott and Storm [SS08] showed $Z_{G}$ determines the number of cycles of length $\ell$ for any $\ell<2 g$.

From (2.3) it is clear that $Z_{G}(t)^{-1}$ is a polynomial in $t$ of degree $2 m^{\dagger}$, where $m^{\dagger}$ is the size of $G^{\dagger}$. In general, though $Z_{G}$ does not determine $m$ or $n$, it does determine $m=m^{\dagger}$ if $G$ is md 2 . If we assume that $G$ is connected (but not necessarily md2), then we can say $Z_{G}$ determines $m-n$. Namely, if $Z_{G}(t)=1$ then $G$ is a tree and $m-n=-1$. Otherwise $G^{\dagger}$ is not the null graph so $m-n=m^{\dagger}-n^{\dagger}$, and $m^{\dagger}$ and $n^{\dagger}$ are determined by $Z_{G}$, as we will see below. Note that $Z_{G}$ does not determine $m-n$ if we drop the assumption that $G$ is connected, since all forests have trivial zeta function.

To get clean statements, for the rest of this section assume $G$ is md 2 and connected. For general connected $G$, the results here can be viewed as results about $G^{\dagger}$ when $G^{\dagger}$ is not the null graph. Note that if $G$ is connected, then $G^{\dagger}$ is also connected.

In this case $m \geq n$, and Hashimoto [Has89] showed (i) $Z_{G}(t)$ has a pole at $t=1$, which is of order $2=m-n+2$ if $G$ is a circuit and $m-n+1$ otherwise, and (ii) $Z_{G}(t)$ has a pole at $t=-1$ of order $m-n, m-n+1$ or $m-n+2$ according to whether $G$ is non-bipartite, a bipartite non-circuit or a bipartite circuit. Hashimoto [Has90] later showed the residue at $t=1$ for non-circuits is $2^{m-n+1}(n-m) \kappa(G)$, where $\kappa$ denotes complexity of $G$, i.e., the number of spanning trees of $G$. Note that $G$ is bipartite if and only if its geodesics all have even length, and recall $G$ is a circuit if and only if $a(\ell)=0$ for $\ell$ sufficiently large.

Consequently, $Z_{G}$ determines $n, m, \kappa(G)$, whether $G$ is bipartite, and whether $G$ is a circuit. In fact, since $\operatorname{det}\left(I-t A+t^{2}(D-I)\right)$ is simply $\operatorname{det}(D+A)=\operatorname{det}|L|$ at $t=-1$, (2.2) tells us $\lim _{t \rightarrow-1}\left(1-t^{2}\right)^{n-m} Z_{G}(t)^{-1}=\operatorname{det}|L|$, which is 0 if and only if $G$ has a bipartite component. So $Z_{G}(t)$ also determines the product of eigenvalues of $|L|$.

Cooper [Coo09] also showed $Z_{G}$ determines whether $G$ is regular, and if so, the degree of regularity as well as the spectrum. In fact if $G$ is $(q+1)$-regular, then the spectrum conversely determines $Z_{G}$ by

$$
\begin{equation*}
Z_{G}(t)=\left(1-t^{2}\right)^{n-m} \prod_{i}\left(1-\lambda_{i} t+q t^{2}\right)^{-1} \tag{2.4}
\end{equation*}
$$

where $\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$ are the eigenvalues of $A$ (attributed to A. Mellein [Cza05]).
Write $V=\left\{v_{1}, \ldots, v_{n}\right\}, d_{i}=\operatorname{deg}\left(v_{i}\right)$ and $q_{i}=d_{i}-1$. We see the leading term of $Z_{G}(t)^{-1}$ is $\left(\prod q_{i}\right) t^{2 m}$. Consequently, for $G \mathrm{md} 2, Z_{G}(t)$ determines $\Pi q_{i}$. One also knows $2 m=\sum d_{i}$, so knowing $Z_{G}(t)$ places rather strong restrictions on the degree sequence for md2 graphs.

In fact, Setyadi-Storm [SS13], based on their enumeration of zeta functions of connected md 2 graphs on $\leq 11$ vertices, conjectured that connected md2 graphs with the same Ihara zeta function have identical degree sequences. However, we give a counterexample.
Example 2.2. The graphs K? ?CA?_FEcdk and K??CA?_ccWNk on 12 vertices and 16 edges drawn below ("the crab and the squid") have the same zeta function, but their degree sequences are (5,5,4, 2, $2,2,2,2,2,2,2,2)$ and $(7,3,3,3,2,2,2,2,2,2,2,2)$.

2.2. Zeta functions of cones-(M1). The most obvious issue of using zeta functions to distinguish arbitrary graphs is that no closed geodesics will pass through "dangling links," i.e., paths not contained in cycles. To resolve this issue, an obvious thing to try is connecting all the degree 1 nodes to a new vertex. However, doing this to just the degree 1 nodes is not a nice operation on graphs (it is not injective), so it makes more sense to look at the cone $G^{*}$ of $G$. This is just the join of $G$ with a point: $G^{*}=G \vee K_{1}$, where $\vee$ denotes the operation of graph join. The new vertex in $G^{*}$ is denoted $v_{n+1}$.

Clearly $G^{*}$ has order $n+1$ and increases the degree of each vertex in $G$ by 1 . That is, $G^{*}$ has degree sequence $\left(d_{i}^{*}\right)$ where $d_{i}^{*}=d_{i}+1$ for $1 \leq i \leq n$ and $d_{n+1}^{*}=n$. Hence if $G$ is an md1 graph (a graph with no degree 0 vertices), then $G^{*}$ is an md2 graph. A graph $H$ on $n+1$ vertices will be the cone of some graph $G$ if and only if there is a vertex of degree $n$, and $G$ can be recovered by deleting any vertex of degree $n$.

We propose to use $Z_{G^{*}}$ to study $G$. Now any edge in $G$ will appear in some closed geodesic in $G^{*}$, so it is reasonable to expect that $Z_{G^{*}}$ encodes much of the structure of $G$. Of course, degree 0 nodes in $G$ become degree 1 nodes in $G^{*}$ and still are not detected by $Z_{G^{*}}$. If one wanted to, say, count the degree 0 nodes only using zeta functions, one could also look at the zeta function of the double cone $G^{* *}=G \vee K_{2}$. Instead, we will simply fix $n$, and see how to determine the number of degree 0 nodes from $n$ and $Z_{G^{*}}$.

Hence, for the rest of this section, we will assume we know the order $n$ of $G$ (except where stated otherwise) and see what can be deduced about $G$ from $Z_{G^{*}}$.

We first show that we can determine the number of degree 0 nodes from $n$ and $Z_{G^{*}}$. Let $G^{\prime}$ be the graph obtained from $G$ by removing all degree 0 nodes. Note $G$ is the empty graph on $n$ vertices if and only if $Z_{G^{*}}=1$. Otherwise, $\left(G^{*}\right)^{\dagger}=\left(G^{\prime}\right)^{*}$ and $\left(G^{\prime}\right)^{*}$ is connected md2, so $Z_{G^{*}}$
determines the number of vertices and edges for $\left(G^{\prime}\right)^{*}$ and hence also for $G^{\prime}$. Combining this with knowing $n$, we can determine $G$ from $G^{\prime}$. Hence in considerations below we may and will assume $G=G^{\prime}$, i.e., $G$ has no degree 0 nodes, thus $G^{*}$ is connected md2.

From the previous section, we immediately see $Z_{G^{*}}$ determines the following: $n, m, \prod d_{i}$, and the number of triangles in $G$, which equals $\frac{1}{2} a_{G^{*}}(3)-m$. The latter follows as a 3-cycle in $G^{*}$ corresponds to either a 3 -cycle in $G$ or a triangle formed from an edge in $G$ with $v_{n+1}$. Similarly, $a_{G^{*}}(4)=a_{G}(4)+2 \sum\binom{d_{i}}{2}$, since the latter term is the number of directed paths of length 2 in $G$. In particular, $Z_{G^{*}}$ determines $a_{G}(4)+\sum d_{i}^{2}$. Also, one knows $a_{G^{*}}(5)$, which is $a_{G}(5)$ plus the number of directed paths of length 3 in $G$.

An elementary consequence of knowing $n, 2 m=\sum d_{i}$ and $\prod d_{i}$ is the following: if we fix $a, b$ and $c$ and know that each vertex has degree $a, b$ or $c$, then $Z_{G^{*}}$ determines the degree sequence. For general $G$, we remark that one can determine the degree sequence by looking at sufficiently many cones. Namely, let $G^{*(r)}=G \vee K_{r}$.
Lemma 2.3. Let $G$ be a graph of possibly unknown order $n \geq 1$. There exists a finite number $r$ such that if $H_{1}, \ldots, H_{r}$ is any known sequence of graphs with distinct orders, then $Z_{G \vee H_{1}}, \ldots, Z_{G \vee H_{r}}$ determines the degree sequence of $G$. One may take $r$ to be at most the order of $G$ (or 3 for $G$ with order less than 3). In particular, $Z_{G^{*}}, \ldots, Z_{G^{*(r)}}$ determines the degree sequence of $G$.

Proof. We will take $r$ to be at least 3, in order to guarantee that one of the $H_{j}$ has order at least 2 (we allow one of the $H_{j}$ 's to be the null graph). We will assume that we know the degree sequences of the $H_{j}$ and describe how the degree sequence of $G$ can be computed from $Z_{G \vee H_{1}}, \ldots, Z_{G \vee H_{r}}$.

Let $H$ be one of the $H_{j}$ 's of order $h \geq 2$. Then $G \vee H$ is connected and if $n \geq 2$ it is also md2. It follows that the order of $(G \vee H)^{\dagger}$ is less than $h+2$ if and only if $n=1$. Otherwise it has order $h+n$ and, in either case, we can find $n$ from $Z_{G \vee H}$. Assume from now on that $n \geq 2$. Then $Z_{G \vee H}$ determines the number of edges of $G \vee H$ from which we can compute the number of edges of $G$, which we will call $m$.

Let $x_{i}$ be the number of vertices of $G$ of degree $i$ for $0 \leq i \leq n-1$. From $n$ and $m$ we have the two linear equations $\sum_{i=0}^{n-1} x_{i}=n$ and $\sum_{i=0}^{n-1} i x_{i}=2 m$. Let $h_{j}$ be the order of $H_{j}$ for $1 \leq j \leq r$. First assume that each $h_{j}$ is at least 2 so $G \vee H_{j}$ are all connected md2 graphs. Then $Z_{G \vee H_{j}}$ determines the products of the degrees of $G \vee H_{j}$ minus 1. The degrees coming from vertices of $H_{j}$ are known and hence $Z_{G \vee H_{j}}$ will determine $\prod_{i=0}^{n-1}\left(i+h_{j}-1\right)^{x_{i}}$, or equivalently, $\sum_{i=0}^{n-1} \ln \left(i+h_{j}-1\right) x_{i}$. Thus we have a linear system with $r+2$ equations and $n$ unknowns. The coefficient matrix $A$ of this linear system has $1, i, \ln \left(i+h_{1}-1\right), \ln \left(i+h_{2}-1\right), \ldots, \ln \left(i+h_{r}-1\right)$ as the entries of its $i$-th column, for $0 \leq i \leq n-1$. If $\mathbf{c}=\left[\begin{array}{llll}c_{1} & c_{2} & \ldots & c_{r+2}\end{array}\right]$ is a vector with $\mathbf{c} A=\mathbf{0}$, then the function $f(t)=c_{1}+c_{2} t+c_{3} \ln \left(t+h_{1}-1\right)+c_{4} \ln \left(t+h_{2}-1\right)+\ldots+c_{r+2} \ln \left(t+h_{r}-1\right)$ has zeros at $0,1,2, \ldots, n-1$. As the $h_{j}$ 's are distinct, this function changes direction at most $r$ times and thus has at most $r+1$ zeros unless all $c_{i}$ are zero. If we take $r=n-2$ then the $c_{i}$ must all be 0 so $A$ is an invertible $n \times n$ matrix and the linear system has exactly one solution. If we drop the assumption that the $h_{j}$ 's are all at least 2 , then at worst we can ignore $Z_{G}$ and $Z_{G^{*}}$ and take $r=n$.
Remark 2.4. If we allow $n \geq 0$ in the above proposition, the order of $G$ can be determined using the $Z_{G \vee H_{j}}$ 's, with one exception: if all the $H_{j}$ 's are empty graphs, then the $Z_{G \vee H_{j}}$ will not distinguish between the null graph and the graph on one vertex. Also, using fewer than $n$ joins often suffices to determine the degree sequence of $G$. For instance, the proof shows $n-2$ works if all $h_{j}$ 's are $\geq 2$. It also ignores the degree information gotten from $Z_{G^{*}}$ if one of the $H_{j}$ has order 1. If we also take into account that the $x_{i}$ must be nonnegative integers, this could significantly decrease the number of joins needed in many cases.

We also comment that for regular graphs, knowing $Z_{G^{*}}$ is equivalent to knowing the spectrum of $G^{*}$ [BS13].
2.3. Zeta functions of graphs and their complements-(M2). We begin by thinking of another way to encode the dangling nodes or links in the zeta function. By a dangling node (link), we mean a vertex (edge) not contained in any cycle, i.e., any node (edge) not in $G^{\dagger}$.

First note that if we fix $n$, we know the number $n-n^{\dagger}$ of dangling nodes, since $Z_{G}$ determines the order $n^{\dagger}$ of $G^{\dagger}$. In particular, $n$ and $Z_{G}$ will tell us if $G$ is connected md2.

Recall that knowing the spectrum of $T$ gives slightly more information than just $Z_{G}$-it also tells us $m$. Since $Z_{G}$ tells us $m^{\dagger}$, knowing $\varphi_{T}$ tell us $m-m^{\dagger}$ which is the number dangling links. However, this tells us nothing about the structure of dangling links-e.g., $\varphi_{T}$ cannot distinguish among a cycle $C_{n}$ with leaves added, $C_{n}$ with a path attached and $C_{n}$ disjoint union a forest, provided the number of edges match.

If we want to somehow account for dangling nodes with zeta functions, we can also try looking at both $G$ and $\bar{G}$. Degree 0 and degree 1 vertices in $G$ now have degree $n-1$ and $n-2$ in $\bar{G}$, so for $n>3$ they will now have degree at least 2. However, it is still possible that some of these vertices are dangling nodes in $\bar{G}$. Nevertheless, we can show the following.

Proposition 2.5. Let $G$ be a graph of order $n$. Then at least one of the following holds:
(i) $G$ is determined by $\varphi_{T}$ and $\varphi_{\bar{T}}$ (or equivalently, by $Z_{G}, Z_{\bar{G}}$ and $m$ ); or
(ii) for any vertex $v \in V$, we have $v \in G^{\dagger}$ or $v \in(\bar{G})^{\dagger}$.

We remark this is not true if (i) is just replaced with " $G$ is determined by $Z_{G}$ and $Z_{\bar{G}}$." For instance, let $G_{1}=K_{3} \sqcup\{v\}$ and $G_{2}$ be the graph obtained from $G_{1}$ by adding an edge from $v$ to one vertex in $K_{3}$. Then $\bar{G}_{1}$ is a tree and $\bar{G}_{2}$ is a forest, so $Z_{G_{1}}=Z_{G_{2}}=Z_{K_{3}}$ and $Z_{\bar{G}_{1}}=$ $Z_{\bar{G}_{2}}=1$. Thus we cannot distinguish $G_{1}$ and $G_{2}$ by looking at their zeta functions and their complements' zeta functions. Moreover, $v$ does not lie in $G_{i}^{\dagger}$ or $\left(\bar{G}_{i}\right)^{\dagger}$ for $i=1,2$.

Before the proof, we give some numerical evidence that (M2)—using $\varphi_{T}$ and $\varphi_{\bar{T}}$ (or $Z_{G}, Z_{\bar{G}}$ and $m$ ) to distinguish graphs of order $n$-is much better than just using $Z_{G}$ and $Z_{\bar{G}}$, or just $Z_{G}$ or $\varphi_{T}$. This data is presented in Table 2.1. The second column is the total number of graphs on $n$ nodes. Subsequent columns contain the number of graphs $G$ on $n$ nodes which are not determined respectively by $Z_{G}$, by $Z_{G}$ and $Z_{\bar{G}}$, by $\varphi_{T}$, and by both $\varphi_{T}$ and $\varphi_{\bar{T}}$.

TAbLE 2.1. Number of small graphs not distinguished by zeta invariants

| $n$ | \# graphs | $Z$ | $Z \bar{Z}$ | $T$ | $T \bar{T}$ |
| :--- | ---: | ---: | ---: | ---: | ---: |
| 2 | 2 | 2 | 2 | 0 | 0 |
| 3 | 4 | 3 | 2 | 0 | 0 |
| 4 | 11 | 8 | 4 | 4 | 0 |
| 5 | 34 | 23 | 8 | 15 | 0 |
| 6 | 156 | 94 | 22 | 75 | 0 |
| 7 | 1,044 | 534 | 68 | 449 | 0 |
| 8 | 12,346 | 4,889 | 312 | 4,297 | 0 |
| 9 | 274,668 | 76,807 | 350 | 68,708 | 2 |

The pair of graphs $\left(G_{1}, G_{2}\right)$ on 9 vertices with the same $\varphi_{T}$ and $\varphi_{\bar{T}}$ are the graphs with 18 edges pictured in Example 2.2. We remark that $G_{1} \simeq \bar{G}_{2}$ in this example.

One explanation for why (M2) is so effective is that in most cases it restricts the problem of distinguishing arbitrary graphs with zeta functions to looking just at connected md2 graphs, where zeta functions give us a lot of information (cf. Section 2.1). Similarly we can determine a
lot about $G$ if $\bar{G}$ is connected md2. And if neither $G$ nor $\bar{G}$ is connected md2, this places strong constraints on $G$, as we will see in the proof.

We first treat a special case, where we can say something stronger.
Lemma 2.6. Suppose $G$ is a forest of order $n$. Then at least one of the following holds:
(i) $\bar{G}$ is connected $m d 2$, or
(ii) $\varphi_{T}$ and $\varphi_{\bar{T}}$ determine $G$ among all graphs of order $n$.

Proof. By Table 2.1, we see $\varphi_{T}$ and $\varphi_{\bar{T}}$ determine $G$ for at least $n \leq 8$. Assume $n \geq 5$. Then $G$ has at most 1 vertex of degree $\geq n-2$. If it has none, $\bar{G}$ is connected md2, so assume it does. Then $G$ must be one of the following three graphs: a star graph on $n$ nodes, a star-like graph on $n$ nodes with exactly 1 node distance 2 from the hub (so the other $n-2$ non-hub nodes are adjacent to the hub), or a star graph on $n-1$ nodes union a point. The latter case can be distinguished from the previous two by counting edges. The former two can be distinguished by looking at the number of edges in $(\bar{G})^{\dagger}$. Since $Z_{G}$ detects forests, $\varphi_{T}$ and $\varphi_{\bar{T}}$ determine these 3 graphs among all graphs of order $n$.
Proof of Proposition 2.5. If $G$ or $\bar{G}$ is md2, then clearly (ii) holds, so we may assume neither is md2. Switching $G$ and $\bar{G}$ if necessary, we may also assume $G$ is connected. In light of the lemma, we may further assume $G^{\dagger}=\left(V^{\dagger}, E^{\dagger}\right)$ is not the null graph, and hence $G^{\dagger}$ has order $n^{\dagger} \geq 3$.

Let $H=(W, F)$ be the subgraph (which is a forest) of $G$ induced from $W=V-V^{\dagger}$. Note that, in $G$, no $w \in W$ can be adjacent to more than 1 vertex in $V^{\dagger}$. Also, in any connected component of $H$, there is exactly one vertex which is adjacent to a vertex in $V^{\dagger}$. Suppose that $H$ contains at least two vertices $w_{1}, w_{2}$. If $w_{1}$ and $w_{2}$ are not adjacent in $G$ then there exists $v \in V^{\dagger}$ such that neither $w_{1}$ nor $w_{2}$ is adjacent to $v$ in $G$ and hence $\bar{G}$ contains the triangle $w_{1}, w_{2}, v$. If $w_{1}$ and $w_{2}$ are adjacent in $G$, then at most one of them is adjacent to a vertex in $V^{\dagger}$ so there exist $v_{1}, v_{2} \in V^{\dagger}$ which are not adjacent to either $w_{1}$ or $w_{2}$. Then $\bar{G}$ contains the 4 -cycle $\left\{w_{1}, w_{2}\right\} \vee\left\{v_{1}, v_{2}\right\}$. In either case, both $w_{1}$ and $w_{2}$ appear in $(\bar{G})^{\dagger}$.

It remains to consider the case where $|W|=1$. Let $w$ be the unique vertex in $W$ and let $v$ be the unique vertex in $V^{\dagger}$ which is adjacent to $w$ in $G$. If any two vertices $v_{1}, v_{2}$ in $V^{\dagger}-\{v\}$ are not adjacent, then $w, v_{1}, v_{2}$ forms a triangle in $\bar{G}$. Also, if there exists $v_{1}, v_{2} \in V^{\dagger}$ which are not adjacent to $v$ then $\{w, v\} \vee\left\{v_{1}, v_{2}\right\}$ is a 4 -cycle in $\bar{G}$. In either case, $w$ appears in $(\bar{G})^{\dagger}$. We may therefore assume the graph induced by $V^{\dagger}-\{v\}$ is a complete graph and there is a most 1 vertex in $V^{\dagger}-\{v\}$ which does not connect to $v$. This leaves us with 2 possibilities for $G$, but we see that $\bar{G}$ for these two possibilities are a star-like graph on $n$ nodes with exactly 1 node distance 2 from the hub or a star graph on $n-1$ nodes union a point. In the proof of the last lemma, we showed that these two graphs are distinguished by $\varphi_{T}$ and $\varphi_{\bar{T}}$ among all graphs of order $n$.

Note that the proofs of the lemma and proposition show that for $n \geq 5$, there are exactly three pairs of graphs $(G, \bar{G})$ of order $n$ for which it is not true that every vertex $v$ of $G$ is in $G^{\dagger}$ or $\bar{G}^{\dagger}$. These are the following three graphs and their complements: the star graph on $n$ nodes, the star-like graph on $n$ nodes with exactly 1 node distance 2 from the hub, and the star graph on $n-1$ nodes union a point. Hence in all but these three cases, every vertex of $G$ appears in a geodesic of $G$ or $\bar{G}$ (or both).

## 3. The Bartholdi zeta function

An alternative to using zeta functions of graphs related to $G$ in order to study $G$ is to consider a more general notion of zeta function which involves dangling nodes and links. If we think
about the adjacency spectrum, or equivalently the closed walk spectrum, it can distinguish things like path graphs from star graphs because backtracking is allowed in closed walks. On the other hand, the closed walk spectrum loses a lot of information contained in the geodesic length spectrum. Bartholdi [Bar99] introduced a more general zeta function which encodes both the closed walk spectrum and the geodesic length spectrum.

Let $\gamma=\left(e_{1}, \ldots, e_{\ell}\right)$ be a closed walk of length $\ell=\ell(\gamma)$. The number of backtracks in $\gamma$ is the number of $1 \leq i<\ell$ such that $e_{i+1}=e_{i}^{-1}$. We say $\gamma$ has a tail if $e_{\ell}=e_{1}^{-1}$. The cyclic bump count $\operatorname{cbc}(\gamma)$ is the number of backtracks in $\gamma$ plus 1 or 0 , according to whether $\gamma$ has a tail or not. The cyclic permutation group $\langle\sigma\rangle$ defined in Section 2 acts on closed walks of length $\ell$ and preserves the cyclic bump count. A closed walk is primitive if it is not of the form $k \gamma$ for $k>1$. Let $a(\ell ; c)=a_{G}(\ell ; c)$ denote the number of $\langle\sigma\rangle$ orbits of primitive closed walks in $G$ with $\ell(\gamma)=\ell$ and $\operatorname{cbc}(\gamma)=c$. Note $a(\ell ; 0)=a(\ell)$ since $\operatorname{cbc}(\gamma)=0$ means $\gamma$ is a geodesic. The Bartholdi zeta function is

$$
\begin{equation*}
\mathcal{Z}_{G}(t, u)=\prod_{\gamma}\left(1-u^{\mathrm{cbc}(\gamma)} t^{\ell(\gamma)}\right)^{-1}=\prod_{c, \ell}\left(1-u^{c} t^{\ell}\right)^{-a(\ell ; c)}=\exp \left(\sum_{c, \ell} a(\ell ; c) \sum_{k \geq 1} \frac{u^{c k} t^{\ell k}}{k}\right), \tag{3.1}
\end{equation*}
$$

where $\gamma$ runs over $\langle\sigma\rangle$ equivalence classes of primitive closed walks in $G$. Note this gives the Ihara zeta function when $u=0: \mathcal{Z}_{G}(t, 0)=Z_{G}(t)$.

Bartholdi [Bar99] proved an analogue of the Bass determinant formula:

$$
\begin{equation*}
\mathcal{Z}_{G}(t, u)=\left(1-(1-u)^{2} t^{2}\right)^{n-m} \operatorname{det}\left(I-t A+(1-u)(D-(1-u) I) t^{2}\right)^{-1} . \tag{3.2}
\end{equation*}
$$

3.1. Properties determined by the Bartholdi zeta function-(M3). First observe (3.1) tells us that knowing $\mathcal{Z}_{G}$ is equivalent to knowing all of the numbers $a(\ell ; c)$. Since $a(2 ; 1)=2 m, \mathcal{Z}_{G}$ determines $m$. However, $\mathcal{Z}_{G}$ does not determine $n$ as adding isolated vertices does not change $\mathcal{Z}_{G}$. We see $\mathcal{Z}_{G}$ determines the number of 3-, 4-, and 5-cycles in $G$ since $Z_{G}$ does. However it does not determine the number of 6 -cycles, as the pair of graphs in Example 2.2 have the same Bartholdi zeta function.
[KL08] observed that one can rewrite (3.2) in terms of the generalized characteristic polynomial $\varphi_{A D}^{G}(\lambda, x)=\operatorname{det}(\lambda I-A+x D)$ by

$$
\begin{equation*}
\mathcal{Z}_{G}(t, u)=\left(1-(1-u)^{2} t^{2}\right)^{n-m} t^{-n} \varphi_{A D}^{G}\left(t^{-1}-(1-u)^{2} t,(1-u) t\right)^{-1}, \tag{3.3}
\end{equation*}
$$

We will write $\varphi_{A D}=\varphi_{A D}^{G}$ if the graph $G$ is clear from context. It is stated in [WLLX11] that $\mathcal{Z}_{G}(t, u)$ determines $\varphi_{A D}$ and vice versa, but this is not true without further qualification in the same way that $Z_{G}$ does not determine $\varphi_{T}$. Namely, $\varphi_{A D}$ determines $m$ and $n$, so also determines $\mathcal{Z}_{G}$. On the other hand, $\mathcal{Z}_{G}$ does not determine $n$. However, since $\mathcal{Z}_{G}$ determines $m$ we see that $\mathcal{Z}_{G}$ and $n$ determine $\varphi_{A D}$ and vice versa.

From now on, we now consider (M3): what can be determined from $n$ and $\mathcal{Z}_{G}$, or equivalently, $\varphi_{A D}$ ?

By specializing $\varphi_{A D}(\lambda, x)$ to $x=0, \pm 1$, we see $\varphi_{A D}$ determines the spectra of $A, L$ and $|L|$. There is much literature about what these spectra individually determine about $G$, and various families of graphs that are determined by such spectra (e.g., see the books [BH12], [CRS10] and the survey articles [vDH03], [vDH09]). We just recall a few things determined by knowing all of these spectra: the number of edges, regularity, the number of components, the number of bipartite components, the complexity and the closed walk spectrum.

From $Z_{G}$, we also know whether $G$ is connected md2, and all of the things discussed in Section 2.1. In addition, [WLLX11] proves that $\varphi_{A D}$ determines the degree sequence of $G$.
3.2. Bartholdi zeta functions of complements-(M4). Finally, consider our last method (M4): what can be determined from $n, \mathcal{Z}_{G}$ and $\mathcal{Z}_{\bar{G}}$.

Equation (3.3) tells us these quantities determine the spectra of $A+x D$ and $A+x D-J$ for all $x$. When $x=0$, [JN80] showed this determines the spectra of $A+y J$ for all $y$, and the proof (see [vDH03] for a simpler proof) in fact works for $x \neq 0$, i.e., $n, \mathcal{Z}_{G}$ and $\mathcal{Z}_{\bar{G}}$ determine the generalized characteristic polynomial $\varphi_{A D J}(\lambda, x, y)=\varphi_{A D J}^{G}(\lambda, x, y)=\operatorname{det}(\lambda I-A+x D+y J)$. The converse, that $\varphi_{A D J}$ determines $n, \mathcal{Z}_{G}$ and $\mathcal{Z}_{\bar{G}}$, is straightforward.

Consequently, (M4) determines everything (M2) and (M3) do. In fact, we show below that (M4) determines everything (M1) does.

Lemma 3.1. Let $X$ and $Y$ be $n \times n$ and $m \times m$ matrices respectively and let $J_{n m}$ denote the $n \times m$ all ones matrix. Let $M$ be the block matrix $M=\left[\begin{array}{cc}X & J_{n m} \\ J_{n m} & Y\end{array}\right]$. We will write $J$ for the square all ones matrix when the order is clear from context. The following are true:
(1) The spectra of $M$ and $J-M$ are determined by the spectra of $X, J-X, Y$, and $J-Y$.
(2) The spectra of $X$ and $J-X$ are determined by the spectra of $M, J-M, Y$, and $J-Y$.

Proof. Note that, for an $r \times r$ matrix, one can determine the spectrum of the matrix from the traces of the first $r$ powers of the matrix and, conversely, one can determine the traces of all powers from the spectrum.

We can prove using induction that $M^{k}$ has the form $\left[\begin{array}{cc}X^{k}+A_{k} & B_{k} \\ C_{k} & Y^{k}+D_{k}\end{array}\right]$ where the matrices $A_{k}, B_{k}, C_{k}, D_{k}$ are sums of matrices which are the product of $k$ matrices coming from $\left\{X, Y, J_{n m}, J_{m n}\right\}$. For $k \geq 2$, each product of $k$ matrices appearing in $A_{k}$ and $D_{k}$ is such that at most $k-2$ of the $k$ matrices are $X^{\prime} \mathrm{s}$ or $Y^{\prime} \mathrm{s}$, and for $B_{k}$ and $C_{k}$ at most $k-1$ are $X^{\prime} \mathrm{s}$ or $Y^{\prime}$ s.

From the equation for $M^{k}$, we get that $\operatorname{tr}\left(M^{k}\right)=\operatorname{tr}\left(X^{k}\right)+\operatorname{tr}\left(Y^{k}\right)+\operatorname{tr}\left(A_{k}\right)+\operatorname{tr}\left(D_{k}\right)$. We also use the following properties: $\operatorname{tr}\left(X J_{n m} Y J_{m n}\right)=\left(\sum_{i, j} x_{i j}\right)\left(\sum_{i, j} y_{i j}\right)=\operatorname{tr}(X J) \operatorname{tr}(Y J)$, trace is invariant under cyclic permutations of products, and the product of any two all ones matrices is a scalar multiple of an all ones matrix. From these properties, it follows that $\operatorname{tr}\left(A_{k}\right)$ and $\operatorname{tr}\left(D_{k}\right)$ can be determined from $\operatorname{tr}\left(X^{i} J\right)$, and $\operatorname{tr}\left(Y^{i} J\right)$ for $i=0,1,2, . ., k-2$. This leads to a relation of the form $\operatorname{tr}\left((J-X)^{k}\right)=(-1)^{k}\left(\operatorname{tr}\left(X^{k}\right)-k \operatorname{tr}\left(X^{k-1} J\right)+\cdots\right)$, where the omitted terms are determined by $\operatorname{tr}\left(X^{i} J\right)$ for $i=0,1,2, \ldots, k-2$, as well as a similar relation for $Y$.

We now prove the two statements. Note that in both cases we can determine $n$ and $m$. Also note that $J-M$ is the block diagonal matrix with $J-X$ and $J-Y$ on the diagonals, so any two of the spectra of $J-M, J-X$, and $J-Y$ determine the third.

First suppose we know the spectra of the matrices $X, J-X, Y$, and $J-Y$. Then we know $\operatorname{tr}\left((J-X)^{i}\right)$ and $\operatorname{tr}\left(X^{i}\right)$ for all $i$. From this, we can recursively determine $\operatorname{tr}\left(X^{i} J\right)$ for all $i$, and similarly for $Y$. We can therefore compute $\operatorname{tr}\left(M^{k}\right)=\operatorname{tr}\left(X^{k}\right)+\operatorname{tr}\left(Y^{k}\right)+\operatorname{tr}\left(A_{k}\right)+\operatorname{tr}\left(D_{k}\right)$ for all $k$ and thus determine the spectrum of $M$.

Suppose now that we know the spectra of $Y, J-Y, M, J-M$. As mentioned above, this tells us the spectrum of $J-X$. It remains to show that we can compute $\operatorname{tr}\left(X^{k}\right)$ for $k=1,2, \ldots, n$. For $k=1,2$ we can find $\operatorname{tr}(X)$ and $\operatorname{tr}\left(X^{2}\right)$ from $\operatorname{tr}(M)=\operatorname{tr}(X)+\operatorname{tr}(Y)$ and $\operatorname{tr}\left(M^{2}\right)=\operatorname{tr}\left(X^{2}\right)+$ $\operatorname{tr}\left(Y^{2}\right)+\operatorname{tr}\left(J_{m n} J_{n m}\right)+\operatorname{tr}\left(J_{n m} J_{m n}\right)$. If $k>2$ and we know $\operatorname{tr}\left(X^{i}\right)$ for $i<k$, then from $\operatorname{tr}\left((J-X)^{i}\right)$ and $\operatorname{tr}\left(X^{i}\right)$ for $i=1,2, . ., k-1$ we can find $\operatorname{tr}\left(X^{i} J\right)$ for $i=1,2, \ldots, k-2$, and therefore also $\operatorname{tr}\left(A_{k}\right)$ and $\operatorname{tr}\left(D_{k}\right)$. From these, we can compute $\operatorname{tr}\left(X^{k}\right)$ from $\operatorname{tr}\left(M^{k}\right)=\operatorname{tr}\left(X^{k}\right)+\operatorname{tr}\left(Y^{k}\right)+\operatorname{tr}\left(A_{k}\right)+$ $\operatorname{tr}\left(D_{k}\right)$.

If $D$ is the degree matrix of $G$, we write $\bar{D}$ for the degree matrix of $\bar{G}$.

Theorem 3.2. Fix $x \in \mathbb{C}$. Let $G_{1}$ and $G_{2}$ be two graphs of the same order and let $H$ be any graph. The graphs $G_{1} \vee H$ and $G_{2} \vee H$ are cospectral with respect to $A+x D$ and $\bar{A}+x \bar{D}$ if and only if $G_{1}$ and $G_{2}$ are cospectral with respect to $A+x D$ and $\bar{A}+x \bar{D}$.
Proof. Let $M=A+x D$ and note that, for an order $r$ graph, $\bar{M}:=\bar{A}+x \bar{D}=J-M+(x(r-$ 1) -1$) I$. Thus if we know $x$ and $r$, knowing the spectrum of $\bar{M}$ is equivalent to knowing the spectrum of $J-M$. Let $n$ be the order of $G_{1}$ and $G_{2}$, let $m$ be the order of $H$, and denote their respective matrices $M=A+x D$ by $M_{G_{i}}$ and $M_{H}$. The join $G_{i} \vee H$ has matrix $M_{G_{i} \vee H}=$ $\left[\begin{array}{cc}M_{G_{i}}+x m I_{n} & J_{n m} \\ J_{m n} & M_{H}+x n I_{m}\end{array}\right]$. We can thus apply the previous lemma to show that if $G_{1}$ and $G_{2}$ are cospectral with respect to $M$ and $\bar{M}$, then so are $G_{1} \vee H$ and $G_{2} \vee H$ and vice versa.

This is already well known in the case that $x=-1$, which corresponds to the Laplacian-in fact less is needed in this case since the Laplacian spectrum of $G$ determines that of $\bar{G}$.

Corollary 3.3. The graphs $G_{1}$ and $G_{2}$ have the same spectra with respect to $A$ and $\bar{A}$ (or $|L|$ and $\overline{|L|})$, if and only if the same is true for $G_{1}^{*}$ and $G_{2}^{*}$. Similarly, $G_{1}$ and $G_{2}$ have the same generalized characteristic polynomial $\varphi_{A D J}$ if and only if the same is true for $G_{1}^{*}$ and $G_{2}^{*}$, i.e., $\varphi_{A D J}^{G_{1}}=\varphi_{A D J}^{G_{2}}$ if and only if $\varphi_{A D J}^{G_{1}^{*}}=\varphi_{A D J}^{G_{2}^{*}}$.

## 4. Constructions

Here we discuss three well-known constructions of cospectral graphs, and show that in many cases, such graphs are distinguished by methods (M1)-(M4) using constraints on degree distributions. This provides some evidence for our conjectures in the next section that most cospectral graphs can be distinguished by any of (M1)-(M4).

In the last part of this section, we give a new construction for graphs which cannot be distinguished by (M1)-(M4), which generalizes the pair of graphs from Example 2.1.
4.1. GM switching. Godsil and McKay present a method for constructing cospectral pairs of graphs [GM82], which is now referred to as GM switching. We say $G$ satisfies the ( $k+1$ )-GM (or just GM) condition if there is an ordering of the vertices such that the adjacency matrix can be written in the form

$$
A=\left[\begin{array}{ccccc}
B_{1} & B_{12} & \ldots & B_{1 k} & N_{1} \\
{ }^{t} B_{12} & B_{2} & \cdots & B_{2 k} & N_{2} \\
\vdots & & \ddots & & \vdots \\
{ }^{t} B_{1 k} & \cdots & { }^{t} B_{k-1, k} & B_{k} & N_{k} \\
{ }^{t} N_{1} & \cdots & { }^{t} N_{k-1} & { }^{t} N_{k} & C
\end{array}\right]
$$

where (i) each $G_{B_{i}}$ is regular, (ii) each $B_{i, j}$ has constant row and column sums, and (iii) each column of each $N_{i}$ has exactly $0, b_{i} / 2$ or $b_{i} 1^{\prime}$ s. Here $b_{i}$ is the order of each square matrix $B_{i}$, and we assume at least one of the $b_{i}{ }^{\prime}$ s is even. Note $k+1$ is the number of diagonal blocks, so the ordering of the vertices and the size of the $B_{i}$ 's determines a partition of the vertex set into $k+1$ subsets $V_{B_{1}}, \ldots, V_{B_{k}}, V_{C}$.

Let $\widetilde{N}_{i}$ be the matrix formed from $N_{i}$ by replacing each column $\mathbf{v}$ of $N_{i}$ which consists of $b_{i} / 2$ ones by the column $J_{b_{i}, 1}-\mathbf{v}$. The GM switch of $G$ is the graph $\widetilde{G}$ with adjacency matrix $\widetilde{A}$ which is gotten from $A$ by replacing each $N_{i}$ with $\widetilde{N}_{i}$. Then $G$ and $\widetilde{G}$ are cospectral with respect to the adjacency matrix. The proof is to exhibit a matrix $Q$ which conjugates $A$ to $\widetilde{A}$ and has constant row and column sums of 1 , and hence commutes with the all ones matrix $J$. It follows that the complements of $G$ and $\widetilde{G}$ are also cospectral with respect to the adjacency matrix. Also, by Corollary 3.3 , the cones of $G$ and $\widetilde{G}$ are cospectral with respect to the adjacency matrix.

Haemers and Spence introduced a special case of GM switching [HS04], which they called $\mathrm{GM}^{*}$ switching, which gives cospectral pairs with respect to any matrix of the form $A+x D$. They only define $\mathrm{GM}^{*}$ switching in the case where $k=1$, but it works in the more general setup as well. We say $G$ satisfies the $(k+1)-\mathrm{GM}^{*}$ condition if $G$ satisfies the $(k+1)$-GM condition with the additional requirement that each vertex in $G_{B_{i}}$ has the same degree in $G$ (i.e., each $N_{i}$ has constant row sums). This condition guarantees that $G$ and $\widetilde{G}$ have the same degree matrix $D$. Also $D$ and $J$ commute with $Q$, the matrix which conjugates $A$ to $\widetilde{A}$, and hence $A+x D+y J$ and $\widetilde{A}+x D+y J$ are cospectral for any $x, y$ so $G$ and $\widetilde{G}$ have the same generalized characteristic polynomial $\varphi_{A D J}$. Also, by Corollary 3.3 , the cones of $G$ and $\widetilde{G}$ have the same $\varphi_{A D J}$. We call this $(k+1)-\mathrm{GM}^{*}$ switching and say that $(G, \widetilde{G})$ is a $(k+1)-\mathrm{GM}^{*}$ pair provided $G \nsucceq \widetilde{G}$.

In the next section, we will find that $\mathrm{GM}^{*}$ switching accounts for a significant percentage of graphs up to 11 vertices which have the same $\varphi_{A D J}$ (see Table 5.1). For the graphs on up to 10 vertices, $3-\mathrm{GM}^{*}$ switching does not produce any such examples that $2-\mathrm{GM}^{*}$ does not. However for 11 vertices, there are $1083-\mathrm{GM}^{*}$ pairs which cannot be obtained by 2-GM* switching, even if one allows successive 2-GM* switching. (It happens that $\left(G_{1}, G_{2}\right)$ can be a $3-\mathrm{GM}^{*}$ pair but not a $2-\mathrm{GM}^{*}$ pair, but that $G_{1}$ and $G_{2}$ are both obtained as 2-GM* switches from a third graph $G_{3}$.)

Example 4.1. The graphs J?BD? X [F[? and J? 'CP ' $\mathrm{LE}\{$ ? on 11 vertices and 18 edges drawn below are a 3-GM* pair, but not a 2-GM* pair. Further, neither of these form a $2-G M^{*}$ pair with any other graph. The subgraphs $G_{B_{1}}, G_{B_{2}}$, and $G_{C}$ are the subgraphs induced by vertices 1-6,7-9, and 10-11 respectively.


We now focus on the case of 2-GM switching. For simplicity, we omit the subscripts and write the adjacency matrix as

$$
A=\left[\begin{array}{cc}
B & N \\
{ }^{N} N & C
\end{array}\right]
$$

For $G \nsucceq \widetilde{G}$, we need $B$ to have even size $\geq 4$. Further, the larger $B$ is, the less likely it is that the GM condition will be satisfied. So Haemers and Spence [HS04, Thm 3] use GM switching with B of size 4 to get a lower bound on the number of non-DS graphs. The following shows that most cospectral pairs thus constructed are distinguished by zeta functions.

Theorem 4.2. Suppose $G_{B}$ is regular on 4 vertices and $G_{C}$ is an md1 (resp. md2) graph on $n$ vertices. Then the proportion of $G M$-admissible choices of $N$ such that the pair $(G, \widetilde{G})$ of labeled graphs formed by GM switching on $(B, C, N)$ which are distinguished by $Z^{*}$ (resp. $Z$ ) goes to 1 as $n \rightarrow \infty$.
Proof. For each column of $N$, there are $2+\binom{4}{2}=8$ possible choices, which we count with equal probability. The probability that at least 2 columns are all ones is

$$
1-\left[\binom{n}{0}\left(\frac{1}{8}\right)^{0}\left(\frac{7}{8}\right)^{n}+\binom{n}{1}\left(\frac{1}{8}\right)^{1}\left(\frac{7}{8}\right)^{n-1}\right]=1-\left(\frac{7+n}{8}\right)\left(\frac{7}{8}\right)^{n-1} .
$$

When this is satisfied, $G$ and $\widetilde{G}$ are md 1 if $G_{C}$ is md 1 and md 2 if $G_{C}$ is md 2 . Since this probability tends to 1 as $n \rightarrow \infty$, from now on, we will assume $G$ and $\widetilde{G}$ are md1 in the case of $Z^{*}$ or md 2 in the case of $Z$.

If $G$ and $\widetilde{G}$ are not distinguished by $Z^{*}$ (resp. $Z$ ), then the products of the vertex degrees (resp. degrees minus 1) must be the same. Since all vertices coming from $G_{C}$ have the same degree, this is equivalent to knowing that the products of the degrees of vertices (resp. degrees minus 1) from $B$ are the same. Let $k$ be the degree of a vertex in $G_{B}$ plus the number of all ones columns in $N$ in the case of $Z^{*}$, or this number minus 1 in the case of $Z$. Let $N_{0}$ be the submatrix of $N$ formed by removing the columns consisting of all zeroes or all ones. Let $x_{i}$ denote the number of 1's in the $i$-th row of $N_{0}$. Then $G$ and $\widetilde{G}$ being distinguished by $Z^{*}$ or $Z$ implies

$$
\begin{equation*}
\prod_{i=1}^{4}\left(k+x_{i}\right)=\prod_{i=1}^{4}\left(k+n_{0}-x_{i}\right) \tag{4.1}
\end{equation*}
$$

where $n_{0}$ is the number of columns in $N_{0}$.
View $n_{0}$ as fixed for now. Since $x_{4}=2 n_{0}-x_{1}-x_{2}-x_{3}$, for fixed $x_{1}, x_{2}$, the solutions to (4.1) in $x_{3}$ are the solutions to a degree 2 polynomial in $x_{3}$, of which there are at most 2 . Now view the top 2 rows of $N_{0}$ as fixed. Let $r$ be the number of columns in $N_{0}$ which have exactly one 1 in the first 2 entries. Then there are $r+1$ choices for $x_{3}$. There are $2^{r}$ choices for row 3 of $N_{0}$, hence for an integer $0 \leq y \leq r$, the probability that $x_{3}=y$ is $\binom{r}{y} \frac{1}{2^{r}} \leq\binom{ r}{[r / 2\rceil} \frac{1}{2^{r}}$. Given $r \geq 3$, the probability that $x_{3}$ avoids solving (4.1) is at least

$$
1-2\binom{r}{\lceil r / 2\rceil} \frac{1}{2^{r}} \geq 1-\frac{2}{\sqrt{3\lceil r / 2\rceil+1}} \geq 1-\sqrt{\frac{8}{3 r}} .
$$

(Here we use the inequality $\binom{2 n}{n} \leq \frac{2^{2 n}}{\sqrt{3 n+1}}$, which implies $\binom{r}{(r / 2\rceil} \leq \frac{2^{r}}{\sqrt{3[r / 27+1}}$.) Also note

$$
P\left(\operatorname{width}\left(N_{0}\right)=n_{0}\right)=\binom{n}{n_{0}}\left(\frac{3}{4}\right)^{n_{0}}\left(\frac{1}{4}\right)^{n-n_{0}}
$$

and for a fixed $n_{0}$ the probability that the number of columns in $N_{0}$ which have exactly one 1 in the first 2 entries is $r$ is

$$
P\left(r \mid n_{0}\right)=\binom{n_{0}}{r}\left(\frac{2}{3}\right)^{r}\left(\frac{1}{3}\right)^{n_{0}-r} .
$$

Fix $0<\delta<1$. By the law of large numbers, for any $\epsilon>0$, the probability that $r \geq n_{0}\left(\frac{2}{3}-\epsilon\right)$ goes to 1 as $n_{0} \rightarrow \infty$. Hence the probability that $r \geq n_{0}^{\delta}$ goes to 1 as $n_{0} \rightarrow \infty$. Similarly, the probability that $n_{0} \geq n^{\delta}$ goes to 1 as $n_{0} \rightarrow \infty$. So the probability that $x_{3}$ avoids the solutions of (4.1) is at least

$$
P\left(n_{0} \geq n^{\delta}\right) P\left(r \geq n_{0}^{\delta}\right)\left(1-\sqrt{\frac{8}{3 n_{0}^{\delta}}}\right) \geq P\left(r \geq n^{\delta^{2}}\right)\left(1-\sqrt{\frac{8}{3 n^{\delta^{2}}}}\right),
$$

which goes to 1 as $n \rightarrow \infty$.
Let $g_{n}$ denote the number of simple graphs (up to isomorphism) of order $n$.
Corollary 4.3. The number of graphs $G$ on $n$ vertices for which there exists a cospectral nonisomorphic graph $\widetilde{G}$ but $Z_{G} \neq Z_{\widetilde{G}}$ is at least $n^{3} g_{n-1}\left(\frac{1}{24}-o(1)\right)$. The same statement is true with $Z_{G} \neq Z_{\widetilde{G}}$ replaced by $Z_{G^{*}} \neq Z_{\widetilde{G}^{*}}$.

Proof. Haemers and Spence [HS04, Thm 3] show there are at least $n^{3} g_{n-1}\left(\frac{1}{24}-o(1)\right)$ nonisomorphic pairs $(G, \widetilde{G})$ obtained from $G M$ switching with $B$ of size 4. Further, for almost all of these pairs, both graphs are md2. Since, in the argument above, we can replace the condition that $C$ is md2 or md1 with knowing $G$ and $\widetilde{G}$ are md2 or md1, we see almost all such pairs are distinguished by $Z$ or $Z^{*}$.
4.2. Coalescence construction. Suppose $G_{1}$ and $G_{2}$ are two graphs of order $n$ with the same adjacency spectra but different degree sequences. Assume there are vertices $x_{1}$ of $G_{1}$ and $x_{2}$ of $G_{2}$ such that $G_{1}-\left\{x_{1}\right\}$ and $G_{2}-\left\{x_{2}\right\}$ are cospectral. Let $U_{i}$ be the vertex set of $G_{i}-\left\{x_{i}\right\}$. Let $\Gamma$ be any graph with a fixed vertex $y$. Let $G_{i}^{\prime}$ be the coalescence of $\left(G_{i}, x_{i}\right)$ with ( $\Gamma, y$ ), i.e., the union of $G_{i}$ and $\Gamma$ after identification of $x_{i}$ with $y$.

Proposition 4.4. With notation as above, $\operatorname{Spec}\left(G_{1}^{\prime}\right)=\operatorname{Spec}\left(G_{2}^{\prime}\right)$ for any $(\Gamma, y)$. If $G_{1}$ and $G_{2}$ are $m d 2$ and $\prod_{v \in U_{2}}(\operatorname{deg}(v)-1) \neq \prod_{v \in U_{2}}(\operatorname{deg}(v)-1)$, then $Z_{G_{1}^{\prime}} \neq Z_{G_{2}^{\prime}}$. Similarly, if $G_{1}$ and $G_{2}$ are md1 and $\prod_{v \in U_{1}} \operatorname{deg}(v) \neq \prod_{v \in U_{2}} \operatorname{deg}(v)$, then $Z_{G_{1}^{\prime *}} \neq Z_{G_{2}^{\prime *}}$.
Proof. The cospectrality is due to Schwenk [Sch73].
For the distinction by zeta functions, note that the fact that $G_{1}-\left\{x_{1}\right\}$ and $G_{2}-\left\{x_{2}\right\}$ are cospectral implies that $\operatorname{deg}_{G_{1}}\left(x_{1}\right)=\operatorname{deg}_{G_{2}}\left(x_{2}\right)$ as the spectrum determines the number of edges. Note $\operatorname{deg}_{G_{i}}(v)=\operatorname{deg}_{G_{i}^{\prime}}(v)$ for any $v \in U_{i}$ and $\operatorname{deg}_{G_{1}^{\prime}}(v)=\operatorname{deg}_{G_{2}^{\prime}}(v)$ for any vertex $v$ of $\Gamma$. The assertions follow as $Z_{G}$ (resp. $Z_{G^{*}}$ ) determines the product of the degrees minus 1 (degrees) of the pruned graph $G^{\dagger}$.
Example 4.5. Let $G_{1}$ and $G_{2}$ be the graphs F ? zPw and F@Rfo on 7 vertices and 10 edges pictured below, where $x_{1}$ and $x_{2}$ are the white vertices.


Then $G_{1}$ and $G_{2}$ are cospectral but not isomorphic, whereas $G_{1}-x_{1}$ and $G_{2}-x_{2}$ are cospectral because they are isomorphic. Here $x_{1}$ and $x_{2}$ have degree 2. The other vertex degrees are $(4,4,4,2,2,2)$ for $G_{1}$ and $(5,3,3,3,2,2)$ for $G_{2}$. It is clear that the products of the degrees and the products of the degrees minus 1 are different. Thus, for any coalescences $G_{1}^{\prime}$ and $G_{2}^{\prime}$ of $\left(G_{1}, x_{1}\right)$ and $\left(G_{2}, x_{2}\right)$ with any $(\Gamma, y)$, we have $Z_{G_{1}^{\prime}} \neq Z_{G_{2}^{\prime}}$ and $Z_{G_{1}^{\prime *}} \neq Z_{G_{2}^{\prime *}}$.

We assumed that $G_{1}$ and $G_{2}$ are md2 or md1 in this proposition for simplicity, but this is not necessary. What one really needs is a condition on pruned subgraphs of $G_{1}^{\prime}$ and $G_{2}^{\prime}$.

Note the coalescence construction includes the case of a disjoint union. If $H_{1}$ and $H_{2}$ are cospectral, we can take $G_{i}$ to be $H_{i}$ disjoint union a single vertex $x_{i}$. Then $G_{1}$ and $G_{2}$ are cospectral, as are $G_{1}-\left\{x_{1}\right\}=H_{1}$ and $G_{2}-\left\{x_{2}\right\}=H_{2}$ by assumption. Let $\Gamma$ be any graph and $y$ any vertex in $\Gamma$. Then the coalescence of $\left(G_{i}, x_{i}\right)$ with $(\Gamma, y)$ is simply the disjoint union $H_{i} \sqcup \Gamma$ of $H_{i}$ with $\Gamma$. However in the case of disjoint unions, we already know the stronger statement that $Z_{H_{1}} \neq Z_{H_{2}}$ implies $Z_{G_{1}^{\prime}} \neq Z_{G_{2}^{\prime}}$ since zeta functions factor into products over their connected components (though this factorization is not true for the zeta of the cones $Z_{G_{i}^{\prime *}}$ ). E.g., if $H_{1}$ and $H_{2}$ are the unique pair of cospectral graphs on 5 vertices, then $Z_{H_{1} \sqcup \Gamma} \neq Z_{H_{2} \sqcup \Gamma}$ for any $\Gamma$, though $G_{1}$ and $G_{2}$ are not md2.
4.3. Join construction. Suppose $G_{1}$ and $G_{2}$ are two graphs of order $n$ which have the same Laplacian spectra but different degree sequences. Let $\Gamma$ be an arbitrary graph. If $G_{1}$ or $G_{2}$ has
an isolated vertex, then assume $\Gamma$ has at least 2 vertices. Then the joins $G_{1} \vee \Gamma$ and $G_{2} \vee \Gamma$ are connected and md2.

Proposition 4.6. With notation as above, $\operatorname{Spec}_{L}\left(G_{1} \vee \Gamma\right)=\operatorname{Spec}_{L}\left(G_{2} \vee \Gamma\right)$ for any $\Gamma$. However $Z_{G_{1} \vee \Gamma} \neq Z_{G_{2} \vee \Gamma}$ for all but finitely many $\Gamma$. Specifically, there is a finite set $S$ consisting of at most $n-1$ integers such that $Z_{G_{1} \vee \Gamma} \neq Z_{G_{2} \vee \Gamma}$ for any $\Gamma$ whose order $r$ does not lie in $S$.

Proof. The first part is true for any $\Gamma$.
Let $d_{1}, \ldots, d_{n}$ be the degree sequence for $G_{1}$ and $d_{1}^{\prime}, \ldots, d_{n}^{\prime}$ be the degree sequence for $G_{2}$. Let $\delta_{1}, \ldots, \delta_{r}$ be the degree sequence for $\Gamma$. Then the degree sequence for $G_{1} \vee \Gamma$ is $d_{1}+r, \ldots, d_{n}+$ $r, \delta_{1}+n, \ldots, \delta_{r}+n$ and similarly for $G_{2} \vee \Gamma$. Assume $r \geq 2$ so $G_{1} \vee \Gamma$ and $G_{2} \vee \Gamma$ are md2. Consequently $Z_{G_{1} \vee \Gamma}$ determines

$$
\prod\left(d_{i}+r-1\right) \cdot \prod\left(\delta_{j}+n-1\right)
$$

and similarly for $G_{2} \vee \Gamma$. Hence $Z_{G_{1} \vee \Gamma} \neq Z_{G_{2} \vee \Gamma}$ if

$$
\prod\left(d_{i}+r-1\right) \neq \prod\left(d_{i}^{\prime}+r-1\right)
$$

Consider the polynomial

$$
f(x)=\prod\left(d_{i}+x\right)-\prod\left(d_{i}^{\prime}+x\right)
$$

which has degree $<n$. The previous equation holds if and only if $r-1$ is not a root of $f(x)$, which can only happen for at most $n-1$ values of $r$.

Note, one can replace the bound on the size of $S$ by the number of differing vertex degrees (counting multiplicity) with $G_{1}$ and $G_{2}$. Also, this proposition gives a non-algorithmic proof of Lemma 2.3 by showing that any two order $n$ graphs with different degree sequences can by distinguished by the zeta functions of $n$ joins with graphs of distinct orders.
Example 4.7. Let $G_{1}$ and $G_{2}$ be ECZO and EEr_. These are graphs on 6 vertices with the same Laplacian spectra, but $G_{1}$ has degree sequence $(4,2,2,2,2,2)$ and $G_{2}$ has degree sequence $(3,3,3,2,2,1)$.


For any (non-null) graph $\Gamma$, the graphs $G_{1} \vee \Gamma$ and $G_{2} \vee \Gamma$ are automatically md2. The polynomial $f(x)$ in the previous proof is just $f(x)=(4+x)(2+x)^{5}-(3+x)^{3}(2+x)^{2}(1+x)=(2+x)^{2}(2 x+5)$. This has no positive roots, so $Z_{G_{1} \vee \Gamma} \neq Z_{G_{2} \vee \Gamma}$ for any $\Gamma$.

The same conclusion is true if we replace $G_{1}$ and $G_{2}$ by their complements.
Corollary 4.8. There are at least $2 g_{n-6}$ pairs of graphs $\left(G_{1}, G_{2}\right)$ of order $n$ which have the same Laplacian spectra but different Ihara zeta functions.
4.4. A new construction. There is one pair of non-isomorphic md2 graphs on 9 vertices which have the same generalized characteristic polynomial $\varphi_{A D J}$, the pair from Example 2.1. This is the smallest such example. This pair does not occur as the result of GM* switching for the following reason: the matrices $A+x D$ are similar for all $x$, but there is no matrix $P$ such that $P^{-1}\left(A_{1}+x D_{1}\right) P=\left(A_{2}+x D_{2}\right)$ for all $x$, as must be the case for $\mathrm{GM}^{*}$ pairs. (In [SS13], it was mistakenly written that all pairs of connected md2 graphs on $n \leq 11$ vertices with the same zeta function, adjacency spectrum, and Laplacian spectra are obtained by GM* switching, but this is false as this example shows. The second author of that paper informed us that the conclusion of that sentence, "are obtained by GM* switching," should be "have the same $\varphi_{A D J}$. .")

There do however exist matrices $P$ and $R$ such that $\left(A_{1}+x D_{1}\right)(P+x R)=(P+x R)\left(A_{2}+x D_{2}\right)$ and $P+x R$ commutes with $J$ for all $x$ and is invertible for all real valued $x$. Here we prove this example is part of a more general construction of pairs of graphs with the same $\varphi_{A D J}$ which cannot be explained by $\mathrm{GM}^{*}$ switching. The fact that the conjugating matrix $P+x R$ needs to depend on $x$ makes this construction more delicate (and complicated) than $\mathrm{GM}^{*}$ switching (and as far as we know, there are no other such constructions). So we first describe it in a concrete way, and then remark afterwards what are the necessary abstract conditions for this construction to work.

Construction 4.9. Consider adjacency matrices $A_{1}, A_{2}$ of the following form:

$$
A_{1}=\left[\begin{array}{ccccc}
B_{1} & B_{12} & \ldots & B_{1 k} & N_{1} \\
{ }^{t} B_{12} & B_{2} & \cdots & B_{2 k} & N_{2} \\
\vdots & & \ddots & & \vdots \\
{ }^{t} B_{1 k} & \cdots & { }^{t} B_{k-1, k} & B_{k} & N_{k} \\
{ }^{t} N_{1} & \cdots & { }^{t} N_{k-1} & { }^{t} N_{k} & C
\end{array}\right], A_{2}=\left[\begin{array}{ccccc}
B_{1} & { }^{t} B_{12} & \cdots & { }^{t} B_{1 k} & N_{1} \\
B_{12} & B_{2} & \cdots & { }^{t} B_{2 k} & N_{2} \\
\vdots & & \ddots & & \vdots \\
B_{1 k} & \cdots & B_{k-1, k} & B_{k} & N_{k} \\
{ }^{t} N_{1} & \cdots & { }^{t} N_{k-1} & { }^{t} N_{k} & C
\end{array}\right]
$$

Here we allow the $B_{i}$ 's to be chosen from among the following $4 \times 4$ adjacency matrices given in $2 \times 2$ block form by:

$$
\left[\begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}\right],\left[\begin{array}{cc}
0 & I \\
I & w
\end{array}\right],\left[\begin{array}{cc}
0 & J \\
J & 0
\end{array}\right],\left[\begin{array}{cc}
w & 0 \\
0 & w
\end{array}\right],\left[\begin{array}{cc}
w & J \\
J & w
\end{array}\right],
$$

where $w=J-I$. The $B_{i j}$ 's $(i<j)$ are allowed to be arbitrarily chosen from among the following 0-1 matrices with constant row and column sums:

$$
\begin{aligned}
& {\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],\left[\begin{array}{ll}
J & J \\
J & J
\end{array}\right],\left[\begin{array}{ll}
0 & J \\
J & 0
\end{array}\right],\left[\begin{array}{cc}
J & 0 \\
0 & J
\end{array}\right],\left[\begin{array}{cc}
I & 0 \\
0 & I
\end{array}\right],\left[\begin{array}{cc}
w & 0 \\
0 & w
\end{array}\right],\left[\begin{array}{cc}
0 & I \\
w & 0
\end{array}\right],\left[\begin{array}{cc}
0 & w \\
I & 0
\end{array}\right],} \\
& {\left[\begin{array}{cc}
I & J \\
J & I
\end{array}\right],\left[\begin{array}{cc}
w & J \\
J & w
\end{array}\right],\left[\begin{array}{ll}
J & w \\
I & J
\end{array}\right],\left[\begin{array}{cc}
J & I \\
w & J
\end{array}\right],\left[\begin{array}{cc}
I & I \\
w & I
\end{array}\right],\left[\begin{array}{cc}
I & w \\
I & I
\end{array}\right],\left[\begin{array}{cc}
w & I \\
w & w
\end{array}\right],\left[\begin{array}{cc}
w & w \\
I & w
\end{array}\right] .}
\end{aligned}
$$

We allow $C$ to be any $m \times m$ adjacency matrix, and the $N_{i}$ 's are $4 \times m$ matrices such that each column of $N_{i}$ consists of all zeros or all ones.

Then $\varphi_{A D J}^{G_{1}}=\varphi_{A D J}^{G_{2}}$, where $G_{i}$ is the graph with adjacency matrix $A_{i}$.
Proof. Note that as each $B_{i j}$ has constant row and column sums and each $N_{i}$ has constant row sums, $G_{1}$ and $G_{2}$ have the same degree matrix, call this $D$. Consider the following $4 \times 4$ matrices:

$$
Q=\left[\begin{array}{cc}
w & w-I \\
w-I & I
\end{array}\right], S=\frac{1}{2}\left[\begin{array}{cc}
I-w & 0 \\
0 & w-I
\end{array}\right]
$$

Let $P$ be the block diagonal matrix whose first $k$ blocks are the matrix $Q$ and last block is the $m \times m$ identity matrix. Let $R$ be the block diagonal matrix whose first $k$ blocks are the matrix $S$ and last block is the $m \times m$ zero matrix.

One can check that $\left(A_{1}+x D\right)(P+x R)=(P+x R)\left(A_{2}+x D\right)$ for all $x$. Note that $P$ has constant row and column sums of 1 and $R$ has constant row and column sums of 0 so $P+x R$ commutes with $J$ for all $x$. Therefore $\left(A_{1}+x D+y J\right)(P+x R)=(P+x R)\left(A_{2}+x D+y J\right)$ for all $x, y$. Also, $\operatorname{det}(P+x R)=\operatorname{det}(Q+x S)^{k}=\left(-x^{2}+2 x-5\right)^{k}$ which is nonzero for all real $x$. It follows that $A_{1}+x D+y J$ and $A_{2}+x D+y J$ have the same eigenvalues for all $x, y$ so $G_{1}$ and $G_{2}$ have the same $\varphi_{A D J}$.
Example 4.10. The pair of graphs $G_{1}$ and $G_{2}$ from Example 2.1 can be obtained from the above construction as follows. Take $k=2$ and $B_{1}=B_{2}=\left[\begin{array}{cc}0 & I \\ I & w\end{array}\right]$ and $B_{12}=\left[\begin{array}{cc}I & I \\ w & I\end{array}\right]$. Take $C$ to be the $1 \times 1$ zero matrix, $N_{1}$ to be the $4 \times 1$ zero vector and $N_{2}$ to be the $4 \times 1$ all ones vector. The resulting graphs
are the graphs from Example 2.1, which we redraw below with the vertices labeled in the order that they appear in this description.


If in the previous example, we take $B_{1}, B_{2}, B_{12}$ as before but let $C$ be any adjacency matrix and $N_{1}=0$ and $N_{2}=J$ (appropriately sized all zero and all ones matrices respectively), then this construction always results in a non-isomorphic pair of graphs. Hence this construction produces at least $g_{n-8}$ non-isomorphic graphs with the same $\varphi_{A D J}$ on $n$ vertices. On 10 vertices, there are 6 different non-isomorphic pairs of graphs that can be built using this construction.

This construction can be made more general. Given a set of $n \times n$ adjacency matrices $B_{i}$ and a set of $n \times n$ matrices $B_{i j}$ with 0 and 1 entries and constant row and column sums, then we can do an analogous construction if there exist $n \times n$ matrices $Q$ and $S$ with the following properties. First, $Q$ and $S$ should have constant row and column sums of 1 and 0 respectively and have $\operatorname{det}(Q+x S)$ not equal to the zero polynomial. If $D_{i}$ is the degree matrix of $M_{i}$, then we need $Q B_{i}=B_{i} Q, D_{i} S=S D_{i}$, and $D_{i} Q-Q D_{i}=S B_{i}-B_{i} S$ for all $i$. Also, we need $B_{i j} Q=Q^{t} B_{i j}$, ${ }^{t} B_{i j} Q=Q B_{i j}, B_{i j} S=S^{t} B_{i j}$, and ${ }^{t} B_{i j} S=S B_{i j}$. If all of these properties are satisfied, then any $A_{1}$ and $A_{2}$ built in the same manner as in the construction will result in graphs with the same $\varphi_{A D J}$. Note that at least one of the $B_{i j}$ must be non-symmetric for $A_{1}$ and $A_{2}$ to be different, but this does not guarantee that the resulting graphs will be non-isomorphic.

## 5. Calculations and Conjectures

In this section, we present our computational study of comparing graphs via the usual spectral methods ( $A, L$ and $|L|$ ) with methods (M1)-(M4). Our findings, together with the above results, lead us to several formal and informal conjectures.

In [HS04], Haemers and Spence enumerated the number of graphs on $n \leq 11$ vertices which are not DS, not $L$-DS, and not $|L|$-DS (see also [BS09] and [Spe] for some errata and additional data for $n=12$ ). This data is summarized in the $A, L$ and $|L|$ columns of Table 5.1. They also enumerated the number of non-DS graphs explained by $2-G M$ switching and $2-\mathrm{GM}^{*}$ switching. This data is recalled in the 2-GM and 2-GM* columns of Table 5.1. The 2-GM column is a lower bound for the $A$ column, and the $2-\mathrm{GM}^{*}$ will be a lower bound for all other columns in the table.

Our main numerical results are a similar enumeration of the number of graphs on $n \leq 11$ vertices which are not determined by methods (M1)-(M4) (considering these methods separatelyconsidering them in tandem is equivalent to just using (M4)). The $\mathcal{Z}$ column of Table 5.1 gives the number of order $n$ graphs which are not $\mathrm{D} \mathcal{Z}$, i.e., not determined by (M3). (Recall that non-DZ implies non-DS, non- $L$-DS, and non- $|L|$-DS.) It-surprisingly to us-turned out that for $n \leq 11$ being $\mathrm{DZ}^{*}, \mathrm{DZZ}$ and $\mathrm{DZ} \overline{\mathcal{Z}}$ (i.e., not determined by (M1), (M2) or (M4)) are all equivalent. The number of graphs which do not satisfy these conditions is given in the $\mathcal{Z} \overline{\mathcal{Z}}$ column.

This data suggests several things: (i) there is not much difference among (M1)-(M4); (ii) any of (M1)-(M4) seems to do much better than $A, L$ or $|L|$; (iii) most GM-pairs are distinguished by

TABLE 5.1. Counting graphs not determined by spectral invariants (M3) is in the $\mathcal{Z}$ column; (M1), (M2) and (M4) are in the $\mathcal{Z} \overline{\mathcal{Z}}$ column

| $n$ | \# graphs | $A$ | $L$ | $\|L\|$ | $\mathcal{Z}$ | $\mathcal{Z} \overline{\mathcal{Z}}$ | 2-GM | 2-GM ${ }^{*}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\leq 5$ | 51 | 2 | 0 | 6 | 0 | 0 | 0 | 0 |
| 6 | 156 | 10 | 4 | 16 | 0 | 0 | 0 | 0 |
| 7 | 1,044 | 110 | 130 | 102 | 0 | 0 | 40 | 0 |
| 8 | 12,346 | 1,722 | 1,767 | 1,201 | 0 | 0 | 1,054 | 0 |
| 9 | 274,668 | 51,038 | 42,595 | 19,001 | 2 | 2 | 38,258 | 0 |
| 10 | $12,005,168$ | $2,560,606$ | $1,412,438$ | 636,607 | 10,146 | 10,140 | $2,047,008$ | 9,480 |
| 11 | $1,018,997,864$ | $215,331,676$ | $91,274,836$ | $38,966,935$ | $1,353,426$ | $1,353,402$ | $176,895,408$ | $1,297,220$ |

(M1)-(M4); (iv) most graphs not distinguished by (M1)-(M4) are explained by 2-GM* switching. (We note that for $n=11,216$ additional non-D $\mathcal{Z} \overline{\mathcal{Z}}$ graphs are explained by 3-GM* switching, but none for $n \leq 10 .{ }^{1}$ Several more non- $D \mathcal{Z} \overline{\mathcal{Z}}$ graphs for $9 \leq n \leq 11$ may also be explained by Construction 4.9.) These suggestions are congruous with our work in previous sections. We just state one of them as a formal conjecture.

Conjecture 5.1. We have: (i) almost all graphs are $D^{*}$; and (ii) almost all non-DS graphs are $\mathrm{DZ}^{*}$. Further, (iii) if $\mathcal{H}_{n}$ (resp. $\mathcal{H}_{n}^{\prime}$ ) is the set of non-DS (resp. non-DZ*) graphs of order $n$, then $\# \mathcal{H}_{n}^{\prime} / \# \mathcal{H}_{n} \rightarrow$ 0 as $n \rightarrow \infty$.

The above statements are also true if we replace $D Z^{*}$ with any of $D Z \bar{Z}, D \mathcal{Z}$, and $D \mathcal{Z} \overline{\mathcal{Z}}$.
In light of Haemers' conjecture that almost all graphs are DS, (ii) seems to be much stronger than (i). Note (iii) is strictly stronger than (ii) as not all DS graphs are DZ*. One reason to believe (ii) is that it appears a significant fraction of non-DS graphs are explained by GM switching. Theorem 4.2 suggests that most of these are distinguished by (M1)-(M4). Further, most of the graphs not distinguished by (M1)-(M4) are explained by $\mathrm{GM}^{*}$ switching, which appears to account for just a trivial fraction of graphs obtained by GM switching (see [HSO4]). Note this reasoning does not apply to distinguishing graphs which are not $L$-DS or $|L|$-DS.

In fact, Theorem 4.2 suggests that just the Ihara zeta function $Z_{G}$ by itself might differentiate most pairs of graphs with the same adjacency spectrum. However, if we are just looking at $Z_{G}$, it is more reasonable to restrict to md2 graphs. Let us say an md2 graph is DZ if there is no other md2 graph of the same order with the same Ihara zeta function.

Setyadi-Storm [SS13] enumerated all pairs of connected md2 graphs with the same Ihara zeta function for $n \leq 11$. This does not exactly tell us the number of non-DZ md2 graphs (even among connected graphs), so we did a similar enumeration to Table 5.1 for md2 graphs with $n \leq 10$. The results are in Table 5.2, with the column headings meaning the same things as in Table 5.1, except restricted to md2 graphs. The only new column is the $Z$ column, which is the number of non-DZ md2 graphs on $n$ vertices.

We remark that while computing this data, we discovered some small errors in the tables in [SS13] for $n=11,12$. Namely, in the cases where more than 2 connected md2 graphs have the same Ihara zeta function, [SS13] undercounts the number of pairs. For instance, for $n=10$, the entries in the [SS13] tables should be augmented by 1 for $m=20,21,24,25$. In each of these cases, there is one triple of connected md2 graphs all with the same zeta function.

[^0]TABLE 5.2. Counting md2 graphs not determined by spectral invariants

| $n$ | \# graphs | $A$ | $L$ | $\|L\|$ | $Z$ | $\mathcal{Z}$ | $\mathcal{Z} \overline{\mathcal{Z}}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\leq 6$ | 77 | 0 | 0 | 4 | 0 | 0 | 0 |
| 7 | 510 | 26 | 64 | 37 | 0 | 0 | 0 |
| 8 | 7,459 | 744 | 1,156 | 725 | 2 | 0 | 0 |
| 9 | 197,867 | 32,713 | 31,353 | 13,878 | 6 | 2 | 2 |
| 10 | $9,808,968$ | $1,727,629$ | $1,184,460$ | 535,080 | 10,130 | 10,094 | 10,088 |

This data, together with our work above, suggests that, when restricting to md2 graphs, there is little difference between methods (M1)-(M4) and just using the Ihara zeta function. Hence we are led to the following conjecture.
Conjecture 5.2. Almost all md2 graphs are DZ, and almost all non-DS md2 graphs are DZ. In fact, if $\mathcal{H}_{n}^{(2)}\left(\right.$ resp. $\left.\mathcal{H}_{n}^{\prime(2)}\right)$ is the set of non-DS (resp. non-DZ) md2 graphs of order $n$, then $\# \mathcal{H}_{n}^{\prime(2)} / \# \mathcal{H}_{n}^{(2)} \rightarrow 0$ as $n \rightarrow \infty$.

Since almost all graphs are md2 for $n$ large, this first statement would imply that almost all graphs of order $n$ are determined by their Ihara zeta function.

We performed the above calculations by generating the graphs with fixed $n$ and $m$ using nauty [nauty] in Sage 6.1.1 [Sage]. Then we used Sage to compute det $|L|=\lim _{t \rightarrow-1}(1-$ $\left.t^{2}\right)^{n-m} Z_{G}(-1)^{-1}$ and $\operatorname{det}(4 D+2 A-3 I)=(-3)^{n-m} Z_{G}(-2)^{-1}$ for each $G$, and similarly for $G^{*}$ and $\bar{G}$. By sorting the graphs according to these values for $G^{*}$ or $G$ and $\bar{G}$, we made a list of candidate non-DZ* and non-DZZ̄ graphs. For each of these pairs, we check to see if the corresponding Ihara zeta functions match up. We compute the Ihara zeta functions by first constructing $G^{\dagger}$, then using the Hashimoto determinant formula. This is much faster than using Sage 6.1.1's built-in function to compute $Z_{G}(t)^{-1}$. To conserve memory, for $n=10,11$, we wrote out the data of the special values of the zeta function with the graph6 string to a file for fixed $n$ and $m$, and sorted these files using the Unix sort tool. Then for each pair of graphs with the same $Z_{G}$ and $Z_{\bar{G}}$, we compared their generalized characteristic polynomials $\varphi_{A D J}$. This takes care of (M1), (M2) and (M4). We will return to (M3) momentarily.

This method of making a first pass by checking just 2 values of the zeta function $Z_{G}(-1)$ and $Z_{G}(-2)$ is based on the heuristic idea that it is unlikely that 2 graphs $G_{1}$ and $G_{2}$ will have the same spectra for several different random linear combinations of $A$ and $D$, unless they have the same spectra for all linear combinations of $A$ and $D$. (This is part of the reason we believe the Ihara zeta function has essentially the same distinguishing power as the Bartholdi zeta function.) From some computational experimentation, we discovered that just knowing $n$, $m$ and the 2 values $Z_{G}(-1)$ and $Z_{G}(-2)$ almost always determines $Z_{G}(t)$ (for $n \leq 11$ ).

This heuristic also suggests that using 2 independent spectra to distinguish graphs is much better than a single one. To further test this idea, we enumerated graphs which are not determined by the following sets of spectra: $A ; A$ and $L ; A, L$ and $|L|$; all three of these plus the Ihara zeta function, and finally $\varphi_{A D}$. These numbers are respectively given in the 3rd through 7th columns of Table 5.3.

Note that even though the numbers of graphs not distinguished by $A$ or $L$ or $|L|$ individually are quite large (cf. Table 5.1), the number of graphs not distinguished by combining 2 or more of these shrinks drastically. (We did not compute the numbers for combining $A$ and $|L|$ or $L$ and $|L|$ for all $n \leq 11$, but we expect a similar phenomenon to hold in these cases also. For instance when $n=9$, there are only 8 graphs not distinguished by $A$ and $|L|$, and 4,405 graphs not distinguished by $L$ and $|L|$.) This suggests that, say, using $A$ and either $L$ or $|L|$, to distinguish graphs is closer in effectiveness to using (M1)-(M4) than to using just $A$ or just $L$ or just $|L|$.

TAble 5.3. Counting graphs not determined by combining spectral invariants

| $n$ | \# graphs | $A$ | $A L$ | $A L\|L\|$ | $A L\|L\| Z$ | $\mathcal{Z}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 5 | 51 | 2 | 0 | 0 | 0 | 0 |
| 6 | 156 | 10 | 0 | 0 | 0 | 0 |
| 7 | 1,044 | 110 | 0 | 0 | 0 | 0 |
| 8 | 12,346 | 1,722 | 0 | 0 | 0 | 0 |
| 9 | 274,668 | 51,038 | 82 | 2 | 2 | 2 |
| 10 | $12,005,168$ | $2,560,606$ | 13,948 | 10,718 | 10,150 | 10,146 |
| 11 | $1,018,997,864$ | $215,331,676$ | $1,468,790$ | $1,361,246$ | $1,353,498$ | $1,353,426$ |

The calculations for Table 5.3 were obtained by comparing spectra via spectral moments, and successively sieving out graphs. E.g., once we know all pairs or graphs with the same $A$ - and $L$-spectra, we search through these to see which also have the same $|L|$-spectra. To compute the last column (i.e., (M3)), we looked at all pairs with the same $A$-, $L$-, $|L|$-spectra and Ihara zeta functions, and checked their generalized characteristic polynomials $\varphi_{A D}$.

We conclude with a couple of final remarks about related calculations.
First, Brouwer and Spence [BS09] find that quite large families can have the same adjacency spectrum ( 46 graphs can have the same spectrum on 11 vertices, and this goes up to 128 on 12 vertices). It seems much rarer for a large family of graphs to have the same zeta functions. We found that on 10 vertices there are 4 triples of graphs with the same $\varphi_{A D J}$, and no larger families. On 11 vertices, there are 1,442 triples of graphs with the same $\varphi_{A D J}$, and 192 quadruples, but no larger families.

Second, one might wonder about using cones or complements with the usual spectra. Knowing the Laplacian spectrum of $G$ is equivalent to knowing it for $\bar{G}$ or $G^{*}$, but what about $A$ or $|L|$ ? From Corollary 3.3, we know that knowing the $A$ - or $|L|$-spectrum of $G$ and $\bar{G}$ implies the same for $G^{*}$ (and conversely knowing the $A$ - or $|L|$-spectrum of $G^{*}$ and $\overline{G^{*}}$ implies the same for $G$ ). In Table 5.4, we enumerate the graphs $G$ on $n \leq 10$ vertices which are not determined by the following: spectrum of $G^{*}$, spectra of $G$ and $\bar{G},|L|$-spectrum of $G^{*}$ and $|L|$-spectrum of $G$ and $\bar{G}$, listed respectively in columns 3 through 6 . The data in the $A \bar{A}$ columns is already in [HS04]. We note there has been recent work towards showing almost all graphs are distinguished by the $A$ - and $\bar{A}$-spectra-see, e.g., [Wan13] or [vDH09].

TABLE 5.4. Counting graphs not determined by cones and complements for $A$ and $|L|$

| $n$ | \# graphs | $A^{*}$ | $A \bar{A}$ | $\|L\|^{*}$ | $\|L\| \mid \overline{L \mid}$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 17 | 0 | 0 | 2 | 2 |
| 5 | 34 | 0 | 0 | 4 | 4 |
| 6 | 156 | 0 | 0 | 16 | 16 |
| 7 | 1,044 | 44 | 40 | 102 | 102 |
| 8 | 12,346 | 1,194 | 1,166 | 1,139 | 1,139 |
| 9 | 274,668 | 44,120 | 43,811 | 18,748 | 18,748 |
| 10 | $12,005,168$ | $2,423,121$ | $2,418,152$ | 633,232 | 633,226 |

We remark that for $A$ - or $|L|$-spectra, there appears to be little difference between using $G$ and $\bar{G}$ versus $G^{*}$ versus just $G$ (cf. Table 5.1; see [HS04], [Spe] for the $A \bar{A}$ data when $n=11,12$ ). However, unlike the case of $L$-spectra, there is at least some difference. In particular, the $A$ - or $|L|$-spectrum of $G^{*}$ does not always determine that of $G$, but Table 5.4 suggests it usually does (particularly for $|L|$ ).

The coincidence of methods (M1) and (M2) on $n \leq 11$ vertices means that for graphs with $n \leq 11$ vertices of size $m, Z_{G^{*}}$ always determines $Z_{G}$ and $Z_{\bar{G}}$, and vice versa. Consequently, for $n \leq 11$ vertices, knowing $Z_{G^{*}}$ is equivalent to knowing $Z_{(\bar{G})^{*}}$. By analogy with the $A-, L$-, and $|L|$-spectra, Corollary 3.3 suggests that $Z_{G}$ and $Z_{\bar{G}}$ (at least almost) always determine $Z_{G^{*}}$, and Table 5.4 suggests that $Z_{G^{*}}$ almost always determines $Z_{G}$ and $Z_{\bar{G}}$. That is, we expect (M1) and (M2) to be almost always equivalent, but there seems to be no reason to expect this is always the case. Similarly, we have no reason to expect (M2) and (M4) always give the same results.

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[^0]:    ${ }^{1}$ To enumerate graphs obtained by 3-GM* switching, we observe that to get a non-isomorphic switch $\widetilde{G}$ from $G$, one needs at least one of the $B_{i}$ 's to be even of size $\geq 4$, say $B_{1}$. One may also assume $B_{2}$ has size $>1$, and the size of $B_{2}$ is not relatively prime to the size of $B_{1}$. Then for each $G$ which is not distinguished by $\varphi_{A D J}$, one can check if it satisfies the 3-GM* condition by iterating first through all "admissible" choices for $V_{B_{1}}$, then through choices for $V_{B_{2}}$.

