Solutions to Review for Final Exam

1. If A is an invertible matrix $n \times n$ matrix, which of the following must be true?

   (a) $\det(A) = 1$.

   (b) The columns of A are an orthogonal set of nonzero vectors in $\mathbb{R}^n$.

   (c) 0 is not an eigenvalue of A.

   (d) The reduced row echelon form of A is I_n.

   (e) A is diagonalizable.

   (f) The rows of A are a basis for $\mathbb{R}_n$.

   (g) The linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = A\mathbf{v}$ is one-to-one and onto.

   Answer: c, d, f, g

2. Which of the following sets are subspaces of $\mathbb{R}^3$?

   (a) A line in $\mathbb{R}^3$ which does not go through the origin.

   (b) A plane through the origin in $\mathbb{R}^3$.

   (c) The origin.

   (d) A sphere of radius 1 in $\mathbb{R}^3$ centered at the origin.

   (e) A ball of radius 1 in $\mathbb{R}^3$ centered at the origin (this is the sphere and its interior)

   (f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$

   (g) The null space of a 4×3 matrix.

   (h) The solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where A is a fixed 3×3 matrix and $\mathbf{b}$ is a fixed nonzero vector.

   (i) The column space of a 3×5 matrix.

   Answer: b, c, g, i

3. Let A be an $n \times n$ skew symmetric matrix.

(a) Prove that if n is odd, then A is not invertible. Hint: Use determinants.

If A is skew symmetric, then $A^T = -A$. Taking determinants we get $\det(A^T) = \det(-A)$. From the properties of determinants, we know that $\det(A^T) = \det(A)$ and $\det(-A) = (-1)^n \det(A)$ and thus we get the equation $\det(A) = (-1)^n \det(A)$. If n is odd, $(-1)^n = -1$ so $\det(A) = -\det(A)$ which forces $\det(A) = 0$ so A is not invertible.

(b) If n is even, can we determine if A is invertible? If yes, give a proof. If no, find examples which show it could be either invertible or non-invertible.

If n is even, $(-1)^n = 1$ so the equation $\det(A) = (-1)^n \det(A)$ becomes $\det(A) = \det(A)$. This gives us no information, so another thing we can try is looking at the smallest case: $n = 2$. If A is 2×2 and skew symmetric, then A must have the form $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$. This matrix has determinant a^2 which is nonzero when $a \neq 0$ and 0 when $a = 0$. We see that A could go either way - for example if $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then it is not invertible and if $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then it is invertible. In both examples, A is $n \times n$ and skew symmetric with n even so these examples prove that for n even we cannot tell if A is invertible.

4. Let $A = \begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix}$.

(a) Find the RREF of A.

The following row operations will put A into RREF. $r_1 \leftrightarrow r_3$, $-r_1 \to r_1$, $r_2 \leftrightarrow r_4$, $r_3 - 2r_1 \to r_3$, $r_3 - 3r_2 \to r_3$, $r_3 \leftrightarrow r_4$, $r_1 + 3r_3 \to r_1$, $r_2 - 2r_3 \to r_3$.

The resulting matrix is $\begin{bmatrix} 1 & 0 & -6 & 0 & 12 \\ 0 & 1 & 4 & 0 & -10 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) What are the rank and nullity of A?

The rank is 3 and nullity is 2. The rank is the number of leading ones in RREF and the nullity is the number of columns without leading ones in

RREF and the sum of the two equals the number of columns.

(c) Find a basis for the null space of A.

The nullity is 2 so a basis for the null space will contain two vectors. The null space is the same as the solution space of $A\mathbf{x} = \mathbf{0}$ which can be found using the RREF of A. If we label the variables as a, b, c, d, e then we see from the RREF that the system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $a - 6c + 12e = 0, b + 4c - 10e = 0, d + 5e = 0$. Columns 3 and 5 do not contain leading ones so the variables c and e can be anything and the other variables can be written in terms of c and e. In particular $a = 6c - 12e, b = -4c + 10e, d = -5e$. The null space is all vectors of the form $\begin{bmatrix} 6c - 12e \\ -4c + 10e \\ c \\ -5e \\ e \end{bmatrix} = c \begin{bmatrix} 6 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -12 \\ 10 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ and has basis $\left\{ \begin{bmatrix} 6 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 10 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$.

(d) Find a basis for the column space of A.

The rank is 3 so both the column space and row space of A have dimension 3 so the bases in this part and the next part will have size 3. The leading ones in RREF are in columns $1, 2, 4$ so columns $1, 2, 4$ of the original matrix will be a basis for the column space so a basis is $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$.

(e) Find a basis for the row space of A.

The nonzero rows from RREF form a basis for the row space so a basis is $\left\{ \begin{bmatrix} 1 & 0 & -6 & 0 & 12 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 0 & -10 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 5 \end{bmatrix} \right\}$.

(f) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ -1 \\ -1 \end{bmatrix}$. Prove that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find all the solutions to $A\mathbf{x} = \mathbf{b}$.

$\begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -1 \\ -1 \end{bmatrix}$. The other solutions to $A\mathbf{x} = \mathbf{b}$ will look like the particular solution $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ plus solutions to the homogeneous linear system (found in part (c)). Therefore the solutions are all vectors of the form $\begin{bmatrix} 1+6c-12e \\ 1-4c+10e \\ -1+c \\ 1-5e \\ 1+e \end{bmatrix}$.

5. Determine if the following statements are true or false. Give a proof or counterexample.

   (a) If U and W are subspaces of a vector space V and $\dim U < \dim W$, then U is a subspace of W.

   False. For U to be a subspace of W it would have to be contained in W. This does not always have to be the case. For example, take $V = \mathbb{R}^3$ and U to be a 1-dimensional subspace and W to be a 2-dimensional subspace. The 1-dimensional subspaces of $\mathbb{R}^3$ are exactly the lines through the origin and the 2-dimensional subspaces are planes though the origin. Given a line and plane though the origin, it is not true that the line must be on the plane. For example, take U to be the z-axis which is all vectors of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ and take W to be the xy-plane which is all vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

   (b) If S is any orthonormal set in $\mathbb{R}^n$, then S is contained in an orthonormal basis for $\mathbb{R}^n$.

   True. S is orthonormal, so it is also orthogonal and the vectors are all nonzero as they are length 1. Therefore S is a linearly independent set of vectors in $\mathbb{R}^n$. Any linearly independent set is contained in a basis, and hence S is contained in a basis T for $\mathbb{R}^n$. When choosing an order for T, put the vectors from S first. Perform the Gram-Schmidt process on T to make a new orthogonal basis R for $\mathbb{R}^n$. As S was already orthogonal, the process will not change the vectors from S. It will perhaps change the remaining vectors so that the new basis R for $\mathbb{R}^n$ is orthogonal. Divide the new vectors by their lengths to make them length 1 (the vectors in S are already length 1). The resulting basis for $\mathbb{R}^n$ will contain S and be orthonormal.

6. Let S be the following set of vectors in $\mathbb{R}^4$.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(a) Find a subset of S which is a basis for span S.

One way to do this is to construct a matrix whose columns are the vectors in S. Then put the matrix in RREF and the columns with leading ones will correspond to the vectors of S which are the basis for span S. The matrix would be $\begin{bmatrix} 1 & 0 & 3 & -7 & 1 \\ 0 & 2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 6 & 6 & 11 & 0 \end{bmatrix}$. We do not actually need to go all the way to RREF, just far enough that it is clear which columns will have leading ones. Doing the row operations $r_4 - r_1 \to r_4, r_4 - 3r_2 \to r_4, r_4 - r_3 \to r_4$ we get $\begin{bmatrix} 1 & 0 & 3 & -7 & 1 \\ 0 & 2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The leading ones will be in columns $1, 2, 5$ so we take the first, second, and fifth vectors of S and our basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Note that the third and fourth vectors were both linear combinations of the first and second vectors, which was why we could delete them without changing the span.

(b) Does S contain a basis for $\mathbb{R}^4$? Is S contained in a basis for $\mathbb{R}^4$?

S does not contain a basis for $\mathbb{R}^4$ because if it did then the span of S would be all of $\mathbb{R}^4$ but the span of S is only 3-dimensional.

S is not contained in a basis for $\mathbb{R}^4$ because it is too big. Any basis for $\mathbb{R}^4$ will contain exactly 4 vectors so it will not be able to contain a set of 5 vectors.

7. Fix a real number λ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$. Determine if the following sets are subspaces of M_{nn}.

   (a) The set of all $n \times n$ matrices with eigenvalue λ.

   This is not a subspace of M_{nn}. Suppose we try to check closed under addition. If A and B are matrices with eigenvalue λ, then $A\mathbf{v_1} = \lambda\mathbf{v_1}$ for some nonzero vector $\mathbf{v_1}$ and $B\mathbf{v_2} = \lambda\mathbf{v_2}$ for another nonzero vector $\mathbf{v_2}$. Since $\mathbf{v_1}$ and $\mathbf{v_2}$ may not be the same vector, there is no obvious way to prove that λ would have to be an eigenvalue of $A + B$. We therefore look for a counterexample to show that it doesn't have to be.

   Suppose $\lambda = 0$. Then the matrices which have $\lambda = 0$ as an eigenvalue are exactly the matrices for which $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This is the set of matrices which are not invertible. This is not closed under addition as $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in the set of non-invertible matrices but their sum is the identity which is invertible. These two matrices both have eigenvalue 0 but their sum does not. This example shows that this set does not have to be a subspace because it is not necessarily closed under addition.

   Note: When $\lambda \neq 0$, it's still not a subspace, since it does not contain the zero matrix.

   (b) The set of all $n \times n$ matrices with eigenvector $\mathbf{v}$.

   This is a subspace. As $\mathbf{v} \neq \mathbf{0}$, it is an eigenvector of the zero matrix O since $O\mathbf{v} = \mathbf{0} = 0\mathbf{v}$. This set therefore contains the zero matrix so it is nonempty. Next check closed under addition. Suppose A and B both have eigenvector $\mathbf{v}$. Then $A\mathbf{v} = \lambda_1\mathbf{v}$ and $B\mathbf{v} = \lambda_2\mathbf{v}$ for some scalars λ_1, λ_2. Their sum has $(A+B)\mathbf{v} = A\mathbf{v} + B\mathbf{v} = \lambda_1\mathbf{v} + \lambda_2\mathbf{v} = (\lambda_1 + \lambda_2)\mathbf{v}$. Therefore $A+B$ also has eigenvector $\mathbf{v}$, with associated eigenvalue $\lambda_1+\lambda_2$. Next check closed under scalar multiplication. If A has eigenvector $\mathbf{v}$ then $A\mathbf{v} = \lambda_1\mathbf{v}$.

For any real number r, $(rA)\mathbf{v} = r(A\mathbf{v}) = r(\lambda_1\mathbf{v}) = (r\lambda_1)\mathbf{v}$ so rA has $\mathbf{v}$ as an eigenvector with associated eigenvalue $r\lambda_1$.

8. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Let W be the subspace of $\mathbb{R}^4$ consisting of vectors which are orthogonal to $\mathbf{v}$.

(a) Find a basis for W and $\dim W$.

A vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in W if and only if $0 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = a + c + d$. Solving $a + c + d = 0$ for a we get that $a = -c - d$ so the vectors in W look like $\begin{bmatrix} -c-d \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. These three vectors span W and are linearly independent so the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W and $\dim W = 3$.

Note that your basis may look slightly different if you solved the equation $a + c + d = 0$ for c or d instead of a.

(b) Find an orthonormal basis for W.

Let $\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Use the Gram-Schmidt formulas to find a new orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} - 0\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

$$\mathbf{v_3} = \mathbf{u_3} - \frac{\mathbf{u_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{u_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} - 0\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\0\\-1/2\\1\end{bmatrix}$$

If we want to avoid fractions, we can instead use $\begin{bmatrix}-1\\0\\-1\\2\end{bmatrix}$ for $\mathbf{v_3}$. This gives us the orthogonal basis $\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\2\end{bmatrix}\right\}$ for W. To get an orthonormal basis, divide each vector by its length. This gives us the orthonormal basis $\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{6}\\0\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}\right\}$.

Note that your result will look different if you found a different basis in part (a) or ordered your vectors differently.

(c) Find an orthonormal basis for $\mathbb{R}^4$ which contains the orthonormal basis for W you found in part (b).

The vector $\mathbf{v}$ is orthogonal to all vectors in W, including the vectors in our basis for W from part (b). We can therefore add $\mathbf{v}/\|\mathbf{v}\|$ to get an orthonormal set, which will be linearly independent and size 4 and thus a basis for $\mathbb{R}^4$. The resulting orthonormal basis is

$$\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{6}\\0\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}, \begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right\}.$$

9. Let P be an $n \times n$ matrix whose columns are an orthonormal set in $\mathbb{R}^n$. Show that $P^{-1} = P^T$.

To show that $P^T = P^{-1}$, we just need to show that one of P^TP or PP^T is the identity. We want to use the fact that the columns are orthogonal, so it is easier to look at P^TP. Label the columns of P as $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$. The i, j-th entry of P^TP is $\mathbf{v_i} \cdot \mathbf{v_j}$. As the $\mathbf{v_i}$ are an orthonormal set, this is 0 if $i \neq j$ and 1 if $i = j$ and hence $P^TP = I$ so P^T is the inverse of P.

10. Let A be a fixed $n \times n$ matrix. Define $L : M_{nn} \to M_{nn}$ to be $L(X) = AX - XA$.

    (a) Prove that L is a linear transformation.

    Check the two properties of linear transformations. $L(X + Y) = A(X + Y) - (X + Y)A = AX + AY - XA - YA = (AX - XA) + (AY - YA) = L(X) + L(Y)$ and $L(rX) = A(rX) - (rX)A = r(AX - XA) = rL(X)$ so L is a linear transformation.

    (b) Is L one-to-one? Is L onto?

    L is not one-to-one. To check if it is one-to-one, we need to check if $\ker L = \{\mathbf{0}\}$. The kernel is all matrices X in M_{nn} such that $AX - XA = \mathbf{0}$, or all X with $AX = XA$. Even though we do not know what A is, there is at least one nonzero matrix which is guaranteed to have this property. If we take $X = I$, the $n \times n$ identity, then $AX = AI = A$ and $XA = IA = A$ so I is in the kernel of L. As the kernel contains more than just the zero matrix, the map is not one-to-one.

    L is also not onto. This is because $\dim M_{nn} = \dim \ker L + \dim \text{range } L$. As the kernel has dimension greater than 0, the dimension of the range must be less than the dimension of M_{nn} (which is n^2). Therefore the range cannot be all of M_{nn} so L is not onto.

11. Let $L : P_3 \to M_{22}$ be the linear transformation $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a-c & 2c+d \\ b+d & 2a-b \end{bmatrix}$.

    (a) Find bases for the kernel and range of L.

    To be in the kernel of L, a polynomial $at^3 + bt^2 + ct + d$ must have $a - c = 0, 2c + d = 0, b + d = 0, 2a - b = 0$. This forces $a = c, d = -2c, b = -d = 2c$ and the kernel is all polynomials of the form $ct^3 + 2ct^2 + ct - 2c$ which has basis $\{t^3 + 2t^2 + t - 2\}$.

    Using that $\dim \ker L + \dim \text{range } L = \dim P_3$ we get that the range will have dimension 3. The range is all matrices of the form $\begin{bmatrix} a-c & 2c+d \\ b+d & 2a-b \end{bmatrix} =$

$a\begin{bmatrix}1&0\\0&2\end{bmatrix}+b\begin{bmatrix}0&0\\1&-1\end{bmatrix}+c\begin{bmatrix}-1&2\\0&0\end{bmatrix}+d\begin{bmatrix}0&1\\1&0\end{bmatrix}$. These four matrices span the range but are not a basis for the range as the range only has dimension 3. One of the matrices must be a linear combination of the others. We see that $\begin{bmatrix}0&1\\1&0\end{bmatrix} = (1/2)\begin{bmatrix}1&0\\0&2\end{bmatrix}+1\begin{bmatrix}0&0\\1&-1\end{bmatrix}+(1/2)\begin{bmatrix}-1&2\\0&0\end{bmatrix}$ so we can delete the fourth matrix without changing the span. We thus get that $\left\{\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}0&0\\1&-1\end{bmatrix},\begin{bmatrix}-1&2\\0&0\end{bmatrix}\right\}$ is a basis for the range of L.

Note: In this case, any one of the four matrices that span the range can be written as linear combination of the other three, so we can delete any one of them to get a basis for the range.

(b) Find the representation of L with respect to the bases $S=\{t^3,t^2,t,1\}$ and $T=\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}$.

$L(t^3)=\begin{bmatrix}1&0\\0&2\end{bmatrix}, L(t^2)=\begin{bmatrix}0&0\\1&-1\end{bmatrix}, L(t)=\begin{bmatrix}-1&2\\0&0\end{bmatrix}, L(1)=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. The coordinate vectors with respect to T are $\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ respectively. Putting these together, we get that the representation is $\begin{bmatrix}1&0&-1&0\\0&0&2&1\\0&1&0&1\\2&-1&0&0\end{bmatrix}$.

(c) Let $S'=\{t^3,t^3-t^2,t^3+t^2-t,t^3+t^2+t-1\}$ and let

$$T'=\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}-1&0\\0&0\end{bmatrix}.$$

Find the representation of L with respect to S' and T' two different ways: directly and using transition matrices.

Method 1: To compute the representation directly, we need to plug the vectors from S' into L. $L(t^3)=\begin{bmatrix}1&0\\0&2\end{bmatrix}, L(t^3-t^2)=\begin{bmatrix}1&0\\-1&3\end{bmatrix}, L(t^3+t^2-t)=\begin{bmatrix}2&-2\\1&1\end{bmatrix}, L(t^3+t^2+t-1)=\begin{bmatrix}0&1\\0&1\end{bmatrix}$. We then need to find the coordinate vectors of each of these with respect to T'. We're looking for x,y,z,w such that $\begin{bmatrix}1&0\\0&2\end{bmatrix}=x\begin{bmatrix}1&0\\0&1\end{bmatrix}+y\begin{bmatrix}0&1\\-1&0\end{bmatrix}+z\begin{bmatrix}0&1\\0&0\end{bmatrix}+w\begin{bmatrix}-1&0\\0&0\end{bmatrix}$. Equivalently, we're solving the system $x - w = 1, y + z = 0, -y = 0, x = 2$ which has solution $x = 2, y = 0, z = 0, w = 1$ so the coordinate vector is $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Similarly, for $\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$ we're solving $x - w = 1, y + z = 0, -y = -1, x = 3$ which has solution $x = 3, y = 1, z = -1, w = 2$ so the coordinate vector is $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}$. The coordinate vectors of $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ respectively. Putting these together we get that the representation is

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix}.$$

Method 2: Suppose A is the representation with respect to S and T and B is the representation with respect to S' and T'. Then $B = Q^{-1}AP$ where P is the transition matrix from S' to S and Q is the transition matrix from T' to T. To find P, we need to find the coordinate vectors of the vectors in S' with respect to S. As S is the standard basis for P_3, the coordinate vectors are just the coefficients of the polynomials which are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. So $P \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. The matrix Q^{-1} is the transition matrix from T to T'. This is more work to compute than P. We can either compute Q (the transition matrix from T' to T) and then find its inverse, or we can compute Q^{-1} directly by finding the coordinate vectors with respect to T' for each vector in T. We will take the second approach. The first vector in T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so we are looking for x, y, z, w such that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + w \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that we need $x = y = z = 0, w = -1$ so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

The coordinate vectors of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ will be $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ respectively. This gives us that $Q^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Multiplying out $Q^{-1}AP$ we get the same answer as in method 1.

12. Let A and B be $n \times n$ matrices. Suppose there exists a basis S for $\mathbb{R}^n$ such that all vectors in S are eigenvectors of both A and B. Prove that $AB = BA$.

    As there are n linearly independent eigenvectors of A and of B (the vectors in S), these matrices are both diagonalizable. In fact, since they have a common basis of eigenvectors, the invertible matrix used to diagonalize them is the same. So there exists diagonal matrices D_1 and D_2 and an invertible matrix P such that $D_1 = P^{-1}AP$ and $D_2 = P^{-1}BP$. In particular, the columns of P are the vectors from S. Solving these equations for A and B we get that $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$. Then $AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1}$ and $BA = PD_2P^{-1}PD_1P^{-1} = PD_2D_1P^{-1}$. But D_1, D_2 are diagonal matrices so $D_1D_2 = D_2D_1$ and therefore $PD_1D_2P^{-1} = PD_2D_1P^{-1}$ so $AB = BA$.

13. Let $p(t) = a_nt^n + a_{n-1}t^{n-1} + ... + a_2t^2 + a_1t + a_0$ be a polynomial. Let A be an $n \times n$ matrix. Define $p(A)$ to be the $n \times n$ matrix $a_nA^n + a_{n-1}A^{n-1} + ... + a_2A^2 + a_1A + a_0I$. If $\mathbf{v}$ is an eigenvector of A with associated eigenvalue λ, prove that $\mathbf{v}$ is an eigenvalue of $p(A)$ with associated eigenvalue $p(\lambda)$.

    First we will show that in general, A^k has eigenvector $\mathbf{v}$ with associated eigenvalue λ^k. We know that $A\mathbf{v} = \lambda\mathbf{v}$. Then $A^k\mathbf{v} = A^{k-1}(A\mathbf{v}) = A^{k-1}(\lambda\mathbf{v}) = \lambda(A^{k-1}\mathbf{v}) = \lambda(A^{k-2}(A\mathbf{v})) = \lambda^2A^{k-2}\mathbf{v} = ... = \lambda^k\mathbf{v}$. This gives us the equation $A^k\mathbf{v} = \lambda^k\mathbf{v}$.

    We are now ready to prove the statement.

$$p(A)\mathbf{v} = (a_nA^n + a_{n-1}A^{n-1} + ... + a_2A^2 + a_1A + a_0I)\mathbf{v}$$
$$= a_n(A^n\mathbf{v}) + a_{n-1}(A^{n-1}\mathbf{v}) + ... + a_2(A^2\mathbf{v}) + a_1(A\mathbf{v}) + a_0(I\mathbf{v})$$
$$= a_n(\lambda^n\mathbf{v}) + a_{n-1}(\lambda^{n-1}\mathbf{v}) + ... + a_2(\lambda^2\mathbf{v}) + a_1(\lambda\mathbf{v}) + a_0\mathbf{v}$$
$$= (a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda + a_0)\mathbf{v} = p(\lambda)\mathbf{v}$$

This proves that $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$ so $\mathbf{v}$ is an eigenvalue of $p(A)$ with associated eigenvalue $p(\lambda)$.

14. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Which of the following matrices have $\mathbf{v}$ as an eigenvector? For those that do have $\mathbf{v}$ as an eigenvector, find the associated eigenvalue.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 2 & -1 & 0 \\ 5 & 0 & 0 & 5 \\ 8 & -7 & 6 & 3 \end{bmatrix}$$

You can check if the matrix has eigenvector $\mathbf{v}$ by multiplying the matrix by $\mathbf{v}$ and seeing if the result is a multiple of $\mathbf{v}$.

$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. This matrix does have $\mathbf{v}$ as an eigenvector with associated eigenvalue 0.

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$. This is not a multiple of $\mathbf{v}$ so this matrix does not have $\mathbf{v}$ as an eigenvector.

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 2 & -1 & 0 \\ 5 & 0 & 0 & 5 \\ 8 & -7 & 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. This matrix does have $\mathbf{v}$ as an eigenvector with associated eigenvalue 10.

(b) Find an example of matrix which has $\mathbf{v}$ as an eigenvector with associated eigenvalue 4.

Any matrix where the entries in each row add up to 4 will work. For example $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ -1 & 5 & 3 & -3 \end{bmatrix}$.

(c) Describe the matrices which have $\mathbf{v}$ as an eigenvector.

These are matrices with constant row sums (meaning the sum of the entries in each row is the same for all the rows).

15. Let $L : P_1 \to P_1$ be the linear transformation given by $L(at+b) = (2a+b)t - a$.

(a) Is L invertible? If yes, what is L^{-1}?

We can do this using a representation of L. Let $S = \{t, 1\}$ and let A be the representation of L with respect to S. $L(t) = 2t - 1$ which has coordinate vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $L(1) = t$ which has coordinate vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

The representation A is invertible, so L is also invertible. The inverse of A is $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ and hence the representation of L^{-1} with respect to S is $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$. Thus $[L^{-1}(at+b)]_S = A^{-1}[at+b]_S = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a+2b \end{bmatrix}$. Then $L^{-1}(at+b) = -bt + (a+2b)$.

(b) Find the eigenvalues and eigenvectors of L.

We can again use the representation $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ to do this problem. We first find the eigenvalues and vectors of A. The characteristic polynomial of A is $\det\left(\begin{bmatrix} \lambda - 2 & -1 \\ 1 & \lambda \end{bmatrix}\right) = (\lambda-2)(\lambda)+1 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$. The only eigenvalue of A is 1 (with multiplicity 2). The associated eigenvectors are the nonzero vectors in the null space of $I - A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. This has RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and the eigenvectors are all nonzero vectors of the from $\begin{bmatrix} -y \\ y \end{bmatrix}$. The eigenvalues of L are the same as the eigenvalues of A, so the only eigenvalue of L is 1. The associated eigenvectors are the vectors in P_1 whose coordinates with respect to S look like $\begin{bmatrix} -y \\ y \end{bmatrix}$ so they are nonzero vectors of the form $-yt + y$.

(c) Is L diagonalizable? If yes, find a basis S for P_1 for which the representation of L with respect to S is diagonal.

The matrix A representing L is not diagonalizable as the eigenvalue 1 has multiplicity 2 but the dimension of the eigenspace is only 1. Therefore L is not diagonalizable and we cannot find a basis for P_1 for which the representation is diagonal. Another way to see this is that in order for the representation to be diagonal, the basis vectors need to be eigenvectors but all the eigenvectors are multiples of $-t+1$ and we thus cannot find 2 linearly independent eigenvectors.

16. For each matrix A, find its eigenvalues and a basis for the associated eigenspaces.

(a) $A = \begin{bmatrix} 2 & -6 & 1 \\ 0 & -1 & 0 \\ -2 & 4 & -1 \end{bmatrix}$

$\det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda - 2 & 6 & -1 \\ 0 & \lambda + 1 & 0 \\ 2 & -4 & \lambda + 1 \end{bmatrix}\right) = \lambda(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda = 0$ with multiplicity 1, $\lambda = 1$ with multiplicity 1, and $\lambda = -1$ with multiplicity 1.

When $\lambda = 0$, we get that $\lambda I - A = \begin{bmatrix} -2 & 6 & -1 \\ 0 & 1 & 0 \\ 2 & -4 & -1 \end{bmatrix}$. This reduces to $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The null space of this matrix is the set of solutions to the homogenous linear system $2x + z = 0, y = 0$. The solutions are of the form $\begin{bmatrix} x \\ 0 \\ -2x \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$.

When $\lambda = 1$, we get that $\lambda I - A = \begin{bmatrix} -1 & 6 & -1 \\ 0 & 2 & 0 \\ 2 & -4 & 2 \end{bmatrix}$. The RREF is

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This gives us the system $x + z = 0, y = 0$. The solutions are of the form $\begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

When $\lambda = -1$, $\lambda I - A = \begin{bmatrix} -3 & 6 & -1 \\ 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$. The RREF is $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This gives us the system $x - 2y = 0, z = 0$. The solutions are of the form $\begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$

$\det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda - 3 & 0 & 0 \\ 2 & \lambda - 3 & 2 \\ -2 & 0 & \lambda - 5 \end{bmatrix} \right) = (\lambda - 3)^2(\lambda - 5)$. The eigenvalues are $\lambda = 3$ with multiplicity 2 and $\lambda = 5$ with multiplicity 1.

When $\lambda = 3$ we get $\lambda I - A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ -2 & 0 & -2 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is the system $x + z = 0$ so y, z can be anything and $x = -z$. The solutions have the form $\begin{bmatrix} -z \\ y \\ z \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 5$ we get $\lambda I - A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is the system $x = 0, y + z = 0$. The solutions have the form $\begin{bmatrix} 0 \\ -z \\ z \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(c) $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\det(\lambda I_4 - A) = \det\left(\begin{bmatrix} \lambda-4 & -2 & 0 & 0 \\ -3 & \lambda-3 & 0 & 0 \\ 0 & 0 & \lambda-2 & -5 \\ 0 & 0 & 0 & \lambda-2 \end{bmatrix}\right) = (\lambda-1)(\lambda-2)^2(\lambda-6)$. The eigenvalues are $\lambda = 1$ with multiplicity 1, $\lambda = 2$ with multiplicity 2, and $\lambda = 6$ with multiplicity 1.

When $\lambda = 1$ we get $\lambda I - A = \begin{bmatrix} -3 & -2 & 0 & 0 \\ -3 & -2 & 0 & 0 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. The RREF is $\begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which gives us the system $x + (2/3)y = 0, z = 0, w = 0$ so the solutions are of the form $\begin{bmatrix} -(2/3)y \\ y \\ 0 \\ 0 \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} -2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$. If you don't like fractions, you could instead take the basis $\left\{\begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \end{bmatrix}\right\}$.

When $\lambda = 2$ we $\lambda I - A = \begin{bmatrix} -2 & -2 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The RREF is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is the system $x = 0, y = 0, w = 0$ so the solutions are of the form $\begin{bmatrix} 0 \\ 0 \\ z \\ 0 \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

When $\lambda = 6$ we get $\lambda I - A = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. The RREF is $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is the system $x - y = 0, z = 0, w = 0$ so the solutions are of the form $\begin{bmatrix} y \\ y \\ 0 \\ 0 \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

17. For each of the matrices in the previous problem, determine if A is diagonalizable. If it is diagonalizable, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$ and find A^{100}.

    (a) This matrix is diagonalizable as it has three distinct eigenvalues (i.e. each eigenvalue has multiplicity 1). D will be the diagonal matrix with the eigenvalues of A along the diagonal and P will be the matrix whose columns are the corresponding eigenvectors. It doesn't matter what order we put the eigenvalues in (so there's more than one correct answer for D and P), but we must make sure the ordering of the eigenvectors matches the orders of the eigenvalues. $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. $D = P^{-1}AP$ so $A = PDP^{-1}$ and $A^{100} = PD^{100}P^{-1}$. To compute A^{100} we first need to compute P^{-1}. We do this by starting with the augmented matrix $[P : I]$ and doing row operations until the left side is I and the matrix on the right will be P^{-1}. The row operations $2r_1 + r_3 \to r_3$, $r_2 \leftrightarrow r_3$, $r_2 - 4r_3 \to r_2$, $r_1 + 2r_3 \to r_1$, $r_2 + r_1 \to r_1$ will take $[P : I]$ to $[I : P^{-1}]$. The inverse is $P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. We then multiply everything out to get that

    $$A^{100} = PD^{100}P^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
    $$= \begin{bmatrix} 2 & -4 + 2(-1)^{100} & 1 \\ 0 & (-1)^{100} & 0 \\ -2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 0 \\ -2 & 4 & -1 \end{bmatrix}.$$

    (b) This matrix is diagonalizable. For each eigenvalue, the dimension of the eigenspace matches the multiplicity of the eigenvalue. The basis vectors for each of the eigenspaces give us three linearly independent eigenvalues. $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. To find P^{-1}, we start with $[P : I]$ and do the row operations $-r_1 \to r_1, r_2 + r_3 \to r_2, r_2 - r_1 \to r_2, r_3 - r_1 \to r_3, r_1 \leftrightarrow r_3, r_2 \leftrightarrow r_3$ to get $[I : P^{-1}]$. The inverse is $P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
    We then multiply everything out to get that

    $$A^{100} = PD^{100}P^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{100} & 0 & 0 \\ 3^{100} - 5^{100} & 3^{100} & 3^{100} - 5^{100} \\ 5^{100} - 3^{100} & 0 & 5^{100} \end{bmatrix}.$$

(c) This matrix is not diagonalizable as $\lambda = 2$ is an eigenvalue with multiplicity 2 but the eigenspace only has dimension 1.