Review for Final Exam

1. If A is an invertible matrix $n \times n$ matrix, which of the following must be true?

   (a) $\det(A) = 1$.

   (b) The columns of A are an orthogonal set of nonzero vectors in $\mathbb{R}^n$.

   (c) 0 is not an eigenvalue of A.

   (d) The reduced row echelon form of A is I_n.

   (e) A is diagonalizable.

   (f) The rows of A are a basis for $\mathbb{R}_n$.

   (g) The linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = A\mathbf{v}$ is one-to-one and onto.

2. Which of the following sets are subspaces of $\mathbb{R}^3$?

   (a) A line in $\mathbb{R}^3$ which does not go through the origin.

   (b) A plane through the origin in $\mathbb{R}^3$.

   (c) The origin.

   (d) A sphere of radius 1 in $\mathbb{R}^3$ centered at the origin.

   (e) A ball of radius 1 in $\mathbb{R}^3$ centered at the origin (this is the sphere and its interior)

   (f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$

   (g) The null space of a 4×3 matrix.

   (h) The solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where A is a fixed 3×3 matrix and $\mathbf{b}$ is a fixed nonzero vector.

   (i) The column space of a 3×5 matrix.

3. Let A be an $n \times n$ skew symmetric matrix.

   (a) Prove that if n is odd, then A is not invertible. Hint: Use determinants.

   (b) If n is even, can we determine if A is invertible? If yes, give a proof. If no, find examples which show it could be either invertible or non-invertible.

4. Let $A = \begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix}$.

   (a) Find the RREF of A.

   (b) What are the rank and nullity of A?

   (c) Find a basis for the null space of A.

   (d) Find a basis for the column space of A.

   (e) Find a basis for the row space of A.

   (f) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ -1 \\ -1 \end{bmatrix}$. Prove that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find all the solutions to $A\mathbf{x} = \mathbf{b}$.

5. Determine if the following statements are true or false. Give a proof or counterexample.

   (a) If U and W are subspaces of a vector space V and $\dim U < \dim W$, then U is a subspace of W.

   (b) If S is any orthonormal set in $\mathbb{R}^n$, then S is contained in an orthonormal basis for $\mathbb{R}^n$.

6. Let S be the following set of vectors in $\mathbb{R}^4$.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

   (a) Find a subset of S which is a basis for span S.

   (b) Does S contain a basis for $\mathbb{R}^4$? Is S contained in a basis for $\mathbb{R}^4$?

7. Fix a real number λ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$. Determine if the following sets are subspaces of M_{nn}.

   (a) The set of all $n \times n$ matrices with eigenvalue λ.

   (b) The set of all $n \times n$ matrices with eigenvector $\mathbf{v}$.

8. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Let W be the subspace of $\mathbb{R}^4$ consisting of vectors which are orthogonal to $\mathbf{v}$.

   (a) Find a basis for W and $\dim W$.

   (b) Find an orthonormal basis for W.

   (c) Find an orthonormal basis for $\mathbb{R}^4$ which contains the orthonormal basis for W you found in part (b).

9. Let P be an $n \times n$ matrix whose columns are an orthonormal set in $\mathbb{R}^n$. Show that $P^{-1} = P^T$.

10. Let A be a fixed $n \times n$ matrix. Define $L : M_{nn} \to M_{nn}$ to be $L(X) = AX - XA$.

    (a) Prove that L is a linear transformation.

    (b) Is L one-to-one? Is L onto?

11. Let $L : P_3 \to M_{22}$ be the linear transformation $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a - c & 2c + d \\ b + d & 2a - b \end{bmatrix}$.

    (a) Find bases for the kernel and range of L.

    (b) Find the representation of L with respect to the bases $S = \{t^3, t^2, t, 1\}$ and $T = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

    (c) Let $S' = \{t^3, t^3 - t^2, t^3 + t^2 - t, t^3 + t^2 + t - 1\}$ and let

    $$T' = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

    Find the representation of L with respect to S' and T' two different ways: directly and using transition matrices.

12. Let A and B be $n \times n$ matrices. Suppose there exists a basis S for $\mathbb{R}^n$ such that all vectors in S are eigenvectors of both A and B. Prove that $AB = BA$.

13. Let $p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_2 t^2 + a_1 t + a_0$ be a polynomial. Let A be an $n \times n$ matrix. Define $p(A)$ to be the $n \times n$ matrix $a_n A^n + a_{n-1} A^{n-1} + ... + a_2 A^2 + a_1 A + a_0 I$. If $\mathbf{v}$ is an eigenvector of A with associated eigenvalue λ, prove that $\mathbf{v}$ is an eigenvalue of $p(A)$ with associated eigenvalue $p(\lambda)$.

14. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Which of the following matrices have $\mathbf{v}$ as an eigenvector? For those that do have $\mathbf{v}$ as an eigenvector, find the associated eigenvalue.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 2 & -1 & 0 \\ 5 & 0 & 0 & 5 \\ 8 & -7 & 6 & 3 \end{bmatrix}$$

   (b) Find an example of matrix which has $\mathbf{v}$ as an eigenvector with associated eigenvalue 4.

   (c) Describe the matrices which have $\mathbf{v}$ as an eigenvector.

15. Let $L : P_1 \to P_1$ be the linear transformation given by $L(at+b) = (2a+b)t - a$.

   (a) Is L invertible? If yes, what is L^{-1}?

   (b) Find the eigenvalues and eigenvectors of L.

   (c) Is L diagonalizable? If yes, find a basis S for P_1 for which the representation of L with respect to S is diagonal.

16. For each matrix A, find its eigenvalues and a basis for the associated eigenspaces.

   (a) $A = \begin{bmatrix} 2 & -6 & 1 \\ 0 & -1 & 0 \\ -2 & 4 & -1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

17. For each of the matrices in the previous problem, determine if A is diagonalizable. If it is diagonalizable, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$ and find A^{100}.