Review for Exam 3

1. Let V be a 3-dimensional vector space with bases S and T. Let $\mathbf{v}$ be a vector such that $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[\mathbf{v}]_S$ if $P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

   Using the formula that $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$ we get that $[\mathbf{v}]_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$.

2. P_2 has basis $S = \{1, t, t^2 + t - 2\}$. Find a basis T for P_2 such that the transition matrix from T to S is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$.

   Let $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. The transition matrix has i-th column equal to $[\mathbf{w_i}]_S$. Hence $[\mathbf{w_1}]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ so $\mathbf{w_1} = 1(1) + 0(t) + 1(t^2 + t - 2) = t^2 + t - 1$. Similarly $\mathbf{w_2} = 2(1) + 1(t) + 3(t^2 + t - 2) = 3t^2 + 4t - 4$ and $\mathbf{w_3} = 0(1) + 0(t) - 1(t^2 + t - 2) = -t^2 - t + 2$ so $T = \{t^2 + t - 1, 3t^2 + 4t - 4, -t^2 - t + 2\}$.

3. Let $V = \mathbb{R}^4$ and let S and T be the bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ and

   $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find $Q_{T \leftarrow S}$ and $P_{S \leftarrow T}$.

   Start with $P_{S \leftarrow T}$. To find the columns of P, we need to take each vector from T and find its coordinate vector with respect to S. For the first vector, write $\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\3\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$. This is the linear system $x = 1, 2y = 0, 3z = 0, 4w = 0$ so the solution is $x = 1, y = 0, z = 0, w = 0$ and the coordinate vector is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$. For the second, write $\begin{bmatrix}1\\2\\0\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\3\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$. This has solution $x = 1, y = 1, z = 0, w = 0$ so the coordinate is $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$. For the third vector, $\begin{bmatrix}1\\2\\3\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\3\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$. This has solution $x = 1, y = 1, z = 1, w = 0$ so the coordinate is $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$. Finally, the fourth vector is $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\3\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$. This has solution $x = 1, y = 1, z = 1, w = 1$ so the coordinate is $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. These are the columns transition matrix, so $P_{S\leftarrow T} = \begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$.

For $Q_{T\leftarrow S}$ we do the same process but switch the roles of S and T. To find the columns of Q, we need to take each vector from S and find its coordinate vector with respect to T. For the first vector, write $\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\3\\0\end{bmatrix} + w\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. This is the linear system $x + y + z + w = 1, 2y + 2z + 2w = 0, 3z + 3w = 0, 4w = 0$ so the solution is $x = 1, y = 0, z = 0, w = 0$ and the coordinate vector is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$. For the second, write $\begin{bmatrix}0\\2\\0\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\3\\0\end{bmatrix} + w\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. This has solution $x = -1, y = 1, z = 0, w = 0$ so the coordinate is $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$. For the third vector, $\begin{bmatrix}0\\0\\3\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\3\\0\end{bmatrix} + w\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. This has solution $x = 0, y = -1, z = 1, w = 0$ so the coordinate is $\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$. Finally, the fourth vector is $\begin{bmatrix}0\\0\\0\\4\end{bmatrix} = x\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\0\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\3\\0\end{bmatrix} + w\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. This has solution $x = 0, y = 0, z = -1, w = 1$ so the coordinate is $\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$. These are the columns transition matrix, so $Q_{T\leftarrow S} = \begin{bmatrix}1 & -1 & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 1 & -1\\0 & 0 & 0 & 1\end{bmatrix}$.

Alternatively, you can compute $Q_{T\leftarrow S}$ by taking the inverse of $P_{S\leftarrow T}$.

(b) Compute $Q_{T\leftarrow S}P_{S\leftarrow T}$.

These matrices are inverses, their product is I_4, the 4×4 identity.

(c) Let $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

To find $[\mathbf{v}]_S$, set $\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$. This has solution

$x = 4, y = 2, z = 4/3, w = 1$ so $[\mathbf{v}]_S = \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix}$.

To find $[\mathbf{v}]_T$, set $\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. This has solution

$x = 2, y = 2/3, z = 1/3, w = 1$ so $[\mathbf{v}]_T = \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix}$.

(d) Confirm that $[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T$ and $[\mathbf{v}]_T = Q_{T\leftarrow S}[\mathbf{v}]_S$.

$$P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix} = [\mathbf{v}]_S$$

$$Q_{T\leftarrow S}[\mathbf{v}]_S = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix} = [\mathbf{v}]_T$$

4. Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.

Hint: Take $\mathbf{u}$ and $\mathbf{v}$ to be vectors starting at the same point which give 2 adjacent sides of the parallelogram. Write down formulas for the diagonals in terms of the vectors $\mathbf{u}$ and $\mathbf{v}$. Use dot products to show that the diagonals are perpendicular if and only if $\mathbf{u}$ and $\mathbf{v}$ are the same length.

The two diagonals are $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$. To check when the diagonals are perpendicular, we take the dot product of these two vectors and see when it is 0. Using properties of dot product we get $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}$. The diagonals are perpendicular if and only if this is 0, so if and only if $\mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v}$. These are both positive numbers, so this is true if and only if $\sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ which is the same as $\|\mathbf{u}\| = \|\mathbf{v}\|$. Two adjacent sides of the parallelogram are the same length if and only if the parallelogram is a rhombus.

5. Let $S = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} \right\}$. Let $V = \operatorname{span} S$.

   (a) Find an orthogonal basis T for V.

   Label the vectors in S as $\mathbf{u_1} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix}$ and the vectors in T will be $\mathbf{v_1}, \mathbf{v_2}$. Using the Gram-Schmidt process we get that $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \mathbf{u_2} - \frac{(\mathbf{v_1},\mathbf{u_2})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} = \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} - \frac{20}{10}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$. The resulting orthogonal basis is $T = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$.

   (b) Find a vector in $\mathbb{R}^3$ which is orthogonal to both vectors in S.

   If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is orthogonal to both vectors in S then $0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3a + c$ and $0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} = -5a - 5b + 5c$. We are therefore looking for solutions to the homogeneous linear system $-3a + c = 0, -5a - 5b + 5c = 0$. The solutions to this are that a can be anything, $b = 2a$, $c = 3a$. Any vector of the form $\begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix}$ is orthogonal to both vectors in S, so one possible answer is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(c) If possible, find an orthogonal basis for $\mathbb{R}^3$ which contains T.

There are two vectors in T and any basis for $\mathbb{R}^3$ contains 3 vectors, so we need to find 1 vector to add to T. The vector we add must be orthogonal to both vectors in T. The vector we found in part (b) is orthogonal to both vectors in S and thus to all vectors in V, including those in T (we can also verify directly that it is orthogonal to the vectors in T). The other two vectors in T are orthogonal, so the set $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is orthogonal and contains T. This set is linearly independent because any orthogonal set of nonzero vectors is linearly independent. It is therefore a basis because any set of three linearly independent vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. The set $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is therefore an orthogonal basis for $\mathbb{R}^3$ which contains T.

6. Let S be the basis $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$. S is a basis for $\mathbb{R}^4$.

(a) Use the Gram-Schmidt process to transform S into an orthonormal basis for $\mathbb{R}^4$.

Label the vectors in S as $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$. We start by building an orthogonal basis which we will label $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$. Using the Gram-Schmidt formulas, we get that the vectors in T are:

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{v_1} \cdot \mathbf{u_2}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v_3} = \mathbf{u_3} - \frac{\mathbf{v_1} \cdot \mathbf{u_3}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{u_3}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix} - \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v_4} = \mathbf{u_4} - \frac{\mathbf{v_1} \cdot \mathbf{u_4}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{u_4}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2} - \frac{\mathbf{v_3} \cdot \mathbf{u_4}}{\mathbf{v_3} \cdot \mathbf{v_3}}\mathbf{v_3}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

The resulting orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$.

To get an orthonormal basis, divide each vector by its length. The resulting orthonormal basis is $\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix} \right\}$.

(b) Write the vector $\begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in the basis from part (a).

If $\mathbf{v} = \begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4}$, then $a_i = \mathbf{v_i} \cdot \mathbf{v}$. Thus $a_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = -\frac{1}{\sqrt{6}}$, $a_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = -2$, $a_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \cdot$

$\begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = \frac{11}{\sqrt{2}}$, $a_4 = \begin{bmatrix} 1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = \frac{8}{\sqrt{12}}$. Therefore

$$\begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix} = -\frac{1}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix} - 2 \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \frac{11}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \frac{8}{\sqrt{12}} \begin{bmatrix} 1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}.$$

7. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be the function $L(\mathbf{u}) = \begin{bmatrix} \mathbf{u} \cdot \mathbf{v_1} \\ \mathbf{u} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{u} \cdot \mathbf{v_k} \end{bmatrix}$.

   (a) Prove that L is a linear transformation.

   Let $\mathbf{u}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Using the properties of dot product, $L(\mathbf{u}+\mathbf{w}) = \begin{bmatrix} (\mathbf{u}+\mathbf{w}) \cdot \mathbf{v_1} \\ (\mathbf{u}+\mathbf{w}) \cdot \mathbf{v_2} \\ \vdots \\ (\mathbf{u}+\mathbf{w}) \cdot \mathbf{v_k} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{v_1} + \mathbf{w} \cdot \mathbf{v_1} \\ \mathbf{u} \cdot \mathbf{v_2} + \mathbf{w} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{u} \cdot \mathbf{v_k} + \mathbf{w} \cdot \mathbf{v_k} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{v_1} \\ \mathbf{u} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{u} \cdot \mathbf{v_k} \end{bmatrix} + \begin{bmatrix} \mathbf{w} \cdot \mathbf{v_1} \\ \mathbf{w} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{w} \cdot \mathbf{v_k} \end{bmatrix} =$ $L(\mathbf{u}) + L(\mathbf{w})$. Also $L(r\mathbf{u}) = \begin{bmatrix} (r\mathbf{u}) \cdot \mathbf{v_1} \\ (r\mathbf{u}) \cdot \mathbf{v_2} \\ \vdots \\ (r\mathbf{u}) \cdot \mathbf{v_k} \end{bmatrix} = \begin{bmatrix} r(\mathbf{u} \cdot \mathbf{v_1}) \\ r(\mathbf{u} \cdot \mathbf{v_2}) \\ \vdots \\ r(\mathbf{u} \cdot \mathbf{v_k}) \end{bmatrix} = r \begin{bmatrix} \mathbf{u} \cdot \mathbf{v_1} \\ \mathbf{u} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{u} \cdot \mathbf{v_k} \end{bmatrix} =$ $rL(\mathbf{u})$.

   (b) Find $\dim \ker L$ and $\dim \operatorname{range} L$.

   Start with the range. As S is orthonormal, $L(\mathbf{v_1}) = \begin{bmatrix} \mathbf{v_1} \cdot \mathbf{v_1} \\ \mathbf{v_1} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{v_1} \cdot \mathbf{v_k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

   Similarly, $L(\mathbf{v_2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ..., $L(\mathbf{v_k}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. We see that the set $\{L(\mathbf{v_1}), ..., L(\mathbf{v_k})\}$

is the standard basis for $\mathbb{R}^k$. These vectors are all in the range as they are L applied to a vector in $\mathbb{R}^n$. The range of L is a subspace of $\mathbb{R}^k$ which contains the standard basis for $\mathbb{R}^k$, so it must be all of $\mathbb{R}^k$. Thus $\dim \operatorname{range} L = k$.

The dimension of the kernel can be gotten using the formula that $\dim \ker L + \dim \operatorname{range} L = \dim \mathbb{R}^n$, so $\dim \ker L = n - k$.

(c) Let W be the set of vectors $\mathbf{w}$ in $\mathbb{R}^n$ such that $\mathbf{w} \cdot \mathbf{v_i} = 0$ for all $i = 1, 2, .., k$. Prove that W is a subspace of $\mathbb{R}^n$ of dimension $n - k$.

W is the kernel of L. We proved that kernels of linear transformations are always subspaces of the starting space, hence W is a subspace of $\mathbb{R}^n$ and from the previous part we know that it has dimension $n - k$.

(d) Assume $k < n$. Let $T = \{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_{n-k}}\}$ be an orthonormal basis for W. Prove that the set $R = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}, \mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_{n-k}}\}$ is an orthonormal basis for $\mathbb{R}^n$.

Start by showing that R is orthonormal. We know that $\mathbf{v_i} \cdot \mathbf{v_j} = 0$ for $i \neq j$ by the orthogonality of S. Similarly, $\mathbf{w_i} \cdot \mathbf{w_j} = 0$ for $i \neq j$ by the orthogonality of T. Also $\mathbf{v_i} \cdot \mathbf{w_j} = 0$ for all i, j as the vector $\mathbf{w_j}$ is in W and the vectors in W are orthogonal to $\mathbf{v_i}$. Thus any possible pair of distinct vectors chosen from R will be orthogonal, so R is orthogonal. Also, the vectors are all length 1 as they are taken from S and T which are orthonormal sets. This shows that R is an orthonormal set.

Any orthonormal set of vectors in $\mathbb{R}^n$ is linearly independent (as it is orthogonal and all vectors are nonzero since they are length 1). There are n vectors in R, so R is a basis for $\mathbb{R}^n$.

8. Which of the following maps are linear transformations?

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} ab - c \\ c + 5a \end{bmatrix}$.

This is not a linear transformation. It does not satisfy either of the properties of linear transformations. For example, $L\left(r\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L\left(\begin{bmatrix} ra \\ rb \\ rc \end{bmatrix}\right) =$

$\begin{bmatrix} r^2ab - rc \\ rc + 5ra \end{bmatrix}$ and $rL\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = r\begin{bmatrix} ab - c \\ c + 5a \end{bmatrix} = \begin{bmatrix} rab - c \\ rc + 5ra \end{bmatrix}$. These are not equal so the property $L(r\mathbf{v}) = rL(\mathbf{v})$ is not satisfied.

(b) $L : P_5 \to \mathbb{R}$ defined by $L(p(t)) = \int_0^1 p(t)\ dt$.

L is a linear transformation. If $p(t)$ and $q(t)$ are two polynomials in P_5 and r is a real number, then $L(p(t) + q(t)) = \int_0^1 p(t) + q(t)\ dt = \int_0^1 p(t)\ dt + \int_0^1 q(t)\ dt = L(p(t)) + L(q(t))$ and $L(rp(t)) = \int_0^1 rp(t)\ dt = r\int_0^1 p(t)\ dt = rL(p(t))$.

9. Let $L : \mathbb{R}^4 \to P_2$ be the linear transformation given by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a - b)t^2 + (c + a)t + (b + c)$$

(a) Find a basis for the kernel of L.

The kernel of L is the subspace of $\mathbb{R}^4$ of vectors which whose image under L is $\mathbf{0}$. If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in the kernel of L then $a - b = 0, c + a = 0, b + c = 0$. The first two equations tell us that $b = a$ and $c = -a$ and thus the last equation $b + c = 0$ is automatically satisfied. There is no restriction on the variable d so we get that $\ker L = \left\{\begin{bmatrix} a \\ a \\ -a \\ d \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} =$ $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. The two vectors are linearly independent so this is a two dimensional vector space with basis $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(b) Find a basis for the range of L.

The range of L is all vectors in P_2 of the form $(a-b)t^2+(c+a)t+(b+c) = a(t^2+t)+b(-t^2+1)+c(t+1)$ so the range of L is $\text{span}\{t^2+t, -t^2+1, t+1\}$. These three vectors are not linearly independent as the third one is the sum of the first two so we can delete the third vector without changing the span. Therefore range $L = \text{span}\{t^2+t, -t^2+1, t+1\} = \text{span}\{t^2+t, -t^2+1\}$. These two vectors are linearly independent so range of L has basis $\{t^2+t, -t^2+1\}$.

Note that we found that both kernel and range of L were dimension 2. A good way to double check these dimensions is to check that they satisfy the equation $\dim \ker L + \dim \text{range } L = \dim \mathbb{R}^4$.

(c) Is L one-to-one? Onto? Invertible?

The dimension of the kernel is 2 so the kernel is not just the zero vector and L is not one-to-one. The range has dimension 2 and P_2 has dimension 3 so the range is not all of P_2 and L is not onto. For L to be invertible, it must be both one-to-one and onto but it is neither so it is not invertible.

10. Let $L : V \to V$ be a linear transformation. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for V. Suppose we know the following:

$$L(\mathbf{v_1}) = \mathbf{v_1} + \mathbf{v_3}$$
$$L(\mathbf{v_2}) = \mathbf{v_1} + 2\mathbf{v_2} + 3\mathbf{v_3}$$
$$L(\mathbf{v_3}) = 2\mathbf{v_3}$$

(a) Find $L(2\mathbf{v_1} - \mathbf{v_2})$.

Using the properties of linear transformations, $L(2\mathbf{v_1} - \mathbf{v_2}) = 2L(\mathbf{v_1}) - L(\mathbf{v_2}) = 2(\mathbf{v_1} + \mathbf{v_3}) - (\mathbf{v_1} + 2\mathbf{v_2} + 3\mathbf{v_3}) = \mathbf{v_1} - 2\mathbf{v_2} - \mathbf{v_3}$.

(b) Find the representation of L with respect to S.

From the three equations given in the problem, $[L(\mathbf{v_1})]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[L(\mathbf{v_2})]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $[L(\mathbf{v_3})]_S = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Hence the representation with respect to S is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$.

(c) Prove that L is invertible and find $L^{-1}(\mathbf{v_3})$.

The easiest way to show L is invertible is to show that the representation of L found in part (b) is an invertible matrix. The determinant of this matrix 4 (nonzero), so it is invertible and so is L.

One way find $L^{-1}(\mathbf{v_3})$ is to apply L^{-1} to both sides of the equation $L(\mathbf{v_3}) = 2\mathbf{v_3}$. This gives that $\mathbf{v_3} = L^{-1}(2\mathbf{v_3}) = 2L^{-1}(\mathbf{v_3})$. Dividing by 2 we get that $L^{-1}(\mathbf{v_3}) = \frac{1}{2}\mathbf{v_3}$.

The other way to do this is to find the inverse of the representation found in part (b). The inverse of that matrix is $\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}$. This is the representation of L^{-1} with respect to S so $[L^{-1}(\mathbf{v_3})]_S = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} [\mathbf{v_3}]_S =$ $\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ so $L^{-1}(\mathbf{v_3}) = 0\mathbf{v_1} + 0\mathbf{v_2} + \frac{1}{2}\mathbf{v_3} = \frac{1}{2}\mathbf{v_3}$.

11. Let V and W be finite dimensional real vector spaces and let $L : V \to W$ be a linear transformation. Circle the correct answer to the following two multiple choice questions.

(a) If L is one-to-one, what can we say about $\dim(V)$ and $\dim(W)$?

$\dim(V) \leq \dim(W)$

L is one-to-one so $\dim \ker L = 0$ so the equation $\dim \ker L + \dim \text{range } L = \dim V$ becomes $\dim \text{range } L = \dim V$. But the range of L is a subspace of W so it has dimension less than or equal to the dimension of W, so $\dim V = \dim \text{range } L \leq \dim W$.

(b) If L is onto, what can we say about $\dim(V)$ and $\dim(W)$?

$\dim(V) \geq \dim(W)$

L is onto so $\dim \text{range } L = \dim W$ so the equation $\dim \ker L + \dim \text{range } L = \dim V$ becomes $\dim \ker L + \dim W = \dim V$ and as $\dim \ker L \geq 0$ this shows $\dim V \geq \dim W$.

12. Let $L : P_2 \to P_2$ be the linear transformation $L(p(t)) = tp'(t) + p(0)$.

   (a) Find the matrix representing L with respect to the basis $\{t^2, t, 1\}$.

   First find L evaluated at each basis element. $L(t^2) = 2t^2, L(t) = t, L(1) = 1$. The coordinate vectors of $2t^2, t, 1$ with respect to the given basis are $\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ respectively. These are the columns of the matrix representing L with respect to the given basis so the matrix is $\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

   (b) Is L invertible? If yes, what is $L^{-1}(4t^2 - t + 3)$?

   L is invertible because the matrix representing L is invertible. The matrix representing L^{-1} with respect to the basis $\{t^2, t, 1\}$ will be the inverse of the matrix in part b which is $\begin{bmatrix}1/2&0&0\\0&1&0\\0&0&1\end{bmatrix}$. The vector $4t^2 - t + 3$ has coordinate vector $\begin{bmatrix}4\\-1\\3\end{bmatrix}$ so $L^{-1}(4t^2 - t + 3)$ will have coordinate vector $\begin{bmatrix}1/2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}4\\-1\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$ so $L^{-1}(4t^2 - t + 3) = 2t^2 - t + 3$.

   Note: This problem can also be done by first rewriting $L(p(t)) = tp'(t) + p(0)$ as $L(at^2 + bt + c) = 2at^2 + bt + c$.

13. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x-y\\2y\\y-3x\end{bmatrix}$.

   Let S be the standard basis for $\mathbb{R}^2$ and $S' = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}\right\}$. Let T be the standard basis for $\mathbb{R}^3$ and $T' = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}\right\}$.

   (a) Find the representation of L with respect to

      i. S and T

We first plug the vectors of S into L. $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-3\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\1\end{bmatrix}$. As T is the standard basis, taking the coordinate vectors with respect to T will not change these vectors so the representation is $\begin{bmatrix}1 & -1\\0 & 2\\-3 & 1\end{bmatrix}$.

ii. S' and T

We start by plugging the vectors in S' into L. $L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-1\\4\\-1\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$. As T is the standard basis, taking the coordinate vector with respect to T does not change the vector so the matrix we get is $\begin{bmatrix}-1 & 1\\4 & -2\\-1 & -1\end{bmatrix}$.

iii. S and T'

As in the first part, if we plug the vectors of S into L we get $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-3\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\1\end{bmatrix}$. We now need to find the coordinate vectors of each of these with respect to T'. To find the coordinate vector of $\begin{bmatrix}1\\0\\-3\end{bmatrix}$ we need to find x, y, z such that $\begin{bmatrix}1\\0\\-3\end{bmatrix} = x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\1\end{bmatrix} + z\begin{bmatrix}0\\0\\2\end{bmatrix}$. In other words, we are trying to solve the system of linear equations $x + y = 1, x + 2y = 0, y + 2z = -3$. The solution is $x = 2, y = -1, z = -1$ so the coordinate vector with respect to T' is $\begin{bmatrix}2\\-1\\-1\end{bmatrix}$. Similarly, to find the coordinate vector of $\begin{bmatrix}-1\\2\\1\end{bmatrix}$ we're solving the linear system $x + y = -1, x + 2y = 2, y + 2z = 1$. The solution is $x = -4, y = 3, z = -1$ so the coordinate vector is $\begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}$. Putting together these two columns we get that the representation with respect to S and T' is $\begin{bmatrix} 2 & -4 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}$.

iv. S' and T'

As in the second part, if we plug the vectors of S' into L we get $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. We now need to find the coordinate vectors of each of these with respect to T'. To find the coordinate vector of $\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ we need to solve the system of linear equations $x + y = -1, x + 2y = 4, y + 2z = -1$. The solution is $x = -6, y = 5, z = -3$ so the coordinate vector with respect to T' is $\begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$. Similarly, to find the coordinate vector of $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ we're solving the linear equation $x + y = 1, x + 2y = -2, y + 2z = -1$. The solution is $x = 4, y = -3, z = 1$ so the coordinate vector is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$. Putting together these two columns we get that the representation with respect to S' and T' is $\begin{bmatrix} -6 & 4 \\ 5 & -3 \\ -3 & 1 \end{bmatrix}$.

(b) Find the transition matrix

i. P from S' to S

To find the columns of P, we need to find the S coordinate vectors of each of the vectors in S'. As S is the standard basis, the coordinate vectors are the same as the original vectors so P is just the matrix with columns equal to the vectors in S'. So $P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

ii. P^{-1} from S to S'

We can either compute this by inverting P from the previous part or we can directly compute the transition matrix from S to S'. To compute this directly, we need to take each of the vectors in S and find their coordinate vectors with respect to S'. To find the coordinate vector of $\begin{bmatrix}1\\0\end{bmatrix}$ with respect to S' we need to find x, y such that $\begin{bmatrix}1\\0\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}0\\-1\end{bmatrix}$. So we are solving the linear system $x = 1, 2x - y = 0$ which has solution $x = 1, y = 2$. To find the coordinate vector of $\begin{bmatrix}0\\1\end{bmatrix}$ we need to solve the linear system $x = 0, 2x - y = 1$ which has solution $x = 0, y = -1$. Putting the coordinate vectors in as the columns of P^{-1} we get that $P^{-1} = \begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}$. As a check, you can verify that $PP^{-1} = I_2$. Coincidentally in this case it turns out that $P = P^{-1}$.

iii. Q from T' to T

To find Q we need to take the vectors in T' and find their coordinate vectors with respect to T. T is the standard basis so the coordinate vectors are the same as the original vectors and Q is the matrix whose columns are the vectors of T', $Q = \begin{bmatrix}1 & 1 & 0\\1 & 2 & 0\\0 & 1 & 2\end{bmatrix}$.

iv. Q^{-1} from T to T'

We can either invert the matrix Q from the previous part or compute this directly by finding the T' coordinate vector of each of the vectors in T. To find the coordinate vector of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ with respect to T' we need to solve the system $x + y = 1, x + 2y = 0, y + 2z = 0$. The solution is $x = 2, y = -1, z = 1/2$ so the coordinate vector is $\begin{bmatrix}2\\-1\\1/2\end{bmatrix}$. To find the coordinate vector of $\begin{bmatrix}0\\1\\0\end{bmatrix}$ with respect to T' we need to solve the system $x + y = 0, x + 2y = 1, y + 2z = 0$. The solution is $x = -1, y = 1, z = -1/2$ so the coordinate vector is $\begin{bmatrix}-1\\1\\-1/2\end{bmatrix}$.

To find the coordinate vector of $\begin{bmatrix}0\\0\\1\end{bmatrix}$ with respect to T' we need to solve the system $x + y = 0, x + 2y = 0, y + 2z = 1$. The solution is $x = 0, y = 0, z = 1/2$ so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$. Putting these together we get that $Q^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$. We can check that $QQ^{-1} = I_3$.

(c) Let A be the representation of L with respect to S and T. Compute AP, $Q^{-1}A$, and $Q^{-1}AP$. How to these compare to the other representations you found?

$$AP = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$Q^{-1}A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}$$

$$Q^{-1}AP = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 5 & -3 \\ -3 & 1 \end{bmatrix}.$$

These are the same as the other three representations we found.