Review for Exam 3

1. Let V be a 3-dimensional vector space with bases S and T. Let $\mathbf{v}$ be a vector such that $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[\mathbf{v}]_S$ if $P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

2. P_2 has basis $S = \{1, t, t^2 + t - 2\}$. Find a basis T for P_2 such that the transition matrix from T to S is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$.

3. Let $V = \mathbb{R}^4$ and let S and T be the bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find $Q_{T \leftarrow S}$ and $P_{S \leftarrow T}$.

   (b) Compute $Q_{T \leftarrow S} P_{S \leftarrow T}$.

   (c) Let $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

   (d) Confirm that $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$ and $[\mathbf{v}]_T = Q_{T \leftarrow S}[\mathbf{v}]_S$.

4. Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.

   Hint: Take $\mathbf{u}$ and $\mathbf{v}$ to be vectors starting at the same point which give 2 adjacent sides of the parallelogram. Write down formulas for the diagonals in terms of the vectors $\mathbf{u}$ and $\mathbf{v}$. Use dot products to show that the diagonals are perpendicular if and only if $\mathbf{u}$ and $\mathbf{v}$ are the same length.

5. Let $S = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} \right\}$. Let $V = \text{span } S$.

   (a) Find an orthogonal basis T for V.

   (b) Find a vector in $\mathbb{R}^3$ which is orthogonal to both vectors in S.

   (c) If possible, find an orthogonal basis for $\mathbb{R}^3$ which contains T.

6. Let S be the basis $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$. S is a basis for $\mathbb{R}^4$.

   (a) Use the Gram-Schmidt process to transform S into an orthonormal basis for $\mathbb{R}^4$.

   (b) Write the vector $\begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in the basis from part (a).

7. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be the function $L(\mathbf{u}) = \begin{bmatrix} \mathbf{u} \cdot \mathbf{v_1} \\ \mathbf{u} \cdot \mathbf{v_2} \\ \vdots \\ \mathbf{u} \cdot \mathbf{v_k} \end{bmatrix}$.

   (a) Prove that L is a linear transformation.

   (b) Find $\dim \ker L$ and $\dim \text{range } L$.

   (c) Let W be the set of vectors $\mathbf{w}$ in $\mathbb{R}^n$ such that $\mathbf{w} \cdot \mathbf{v_i} = 0$ for all $i = 1, 2, .., k$. Prove that W is a subspace of $\mathbb{R}^n$ of dimension $n - k$.

   (d) Assume $k < n$. Let $T = \{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_{n-k}}\}$ be an orthonormal basis for W. Prove that the set $R = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}, \mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_{n-k}}\}$ is an orthonormal basis for $\mathbb{R}^n$.

8. Which of the following maps are linear transformations?

   (a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} ab - c \\ c + 5a \end{bmatrix}$.

   (b) $L : P_5 \to \mathbb{R}$ defined by $L(p(t)) = \int_0^1 p(t)\ dt$.

9. Let $L : \mathbb{R}^4 \to P_2$ be the linear transformation given by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a-b)t^2 + (c+a)t + (b+c)$$

(a) Find a basis for the kernel of L.

(b) Find a basis for the range of L.

(c) Is L one-to-one? Onto? Invertible?

10. Let $L : V \to V$ be a linear transformation. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for V. Suppose we know the following:

$$L(\mathbf{v_1}) = \mathbf{v_1} + \mathbf{v_3}$$

$$L(\mathbf{v_2}) = \mathbf{v_1} + 2\mathbf{v_2} + 3\mathbf{v_3}$$

$$L(\mathbf{v_3}) = 2\mathbf{v_3}$$

(a) Find $L(2\mathbf{v_1} - \mathbf{v_2})$.

(b) Find the representation of L with respect to S.

(c) Prove that L is invertible and find $L^{-1}(\mathbf{v_3})$.

11. Let V and W be finite dimensional real vector spaces and let $L : V \to W$ be a linear transformation. Circle the correct answer to the following two multiple choice questions.

(a) If L is one-to-one, what can we say about $\dim(V)$ and $\dim(W)$?

i. $\dim(V) \leq \dim(W)$

ii. $\dim(V) \geq \dim(W)$

iii. $\dim(V) = \dim(W)$

iv. none of the above

(b) If L is onto, what can we say about $\dim(V)$ and $\dim(W)$?

i. $\dim(V) \leq \dim(W)$

ii. $\dim(V) \geq \dim(W)$

iii. $\dim(V) = \dim(W)$

iv. none of the above

12. Let $L : P_2 \to P_2$ be the linear transformation $L(p(t)) = tp'(t) + p(0)$.

(a) Find the matrix representing L with respect to the basis $\{t^2, t, 1\}$.

(b) Is L invertible? If yes, what is $L^{-1}(4t^2 - t + 3)$?

13. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2y \\ y - 3x \end{bmatrix}$.

Let S be the standard basis for $\mathbb{R}^2$ and $S' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$. Let T be the standard basis for $\mathbb{R}^3$ and $T' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$.

(a) Find the representation of L with respect to

i. S and T

ii. S' and T

iii. S and T'

iv. S' and T'

(b) Find the transition matrix

i. P from S' to S

ii. P^{-1} from S to S'

iii. Q from T' to T

iv. Q^{-1} from T to T'

(c) Let A be the representation of L with respect to S and T. Compute AP, $Q^{-1}A$, and $Q^{-1}AP$. How to these compare to the other representations you found?