Review for Exam 2

Note: All vector spaces are real vector spaces. Definition 4.4 will be provided on the exam as it appears in the textbook.

1. Determine if the set V together with operations $\oplus$ and $\odot$ is a vector space. Either show that Definition 4.4 is satisfied or determine which properties of Definition 4.4 fail to hold.

   (a) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = c + \mathbf{u}$.

   V is closed under $\oplus$ and $\odot$ and satisfies properties 1-3 of the definition but fails properties 4-8.

   Closed under $\oplus$ and $\odot$: the product of any two real numbers is a real number and the sum of any two real numbers is a real number.

   Properties 1 and 2: These hold because they hold for multiplication of real numbers.

   Property 3: The number 1 plays the role of the zero vector because $\mathbf{u} \oplus 1 = \mathbf{u}$ for all $\mathbf{u} \in V$.

   Property 4: Because 1 is the zero vector, the negative of a vector $\mathbf{u}$ will be some $\mathbf{a} \in V$ such that $\mathbf{u} \oplus \mathbf{a} = 1$ so we would need $\mathbf{a} = 1/\mathbf{u}$. This property fails because 0 is in V but $1/0$ is not defined.

   Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot (\mathbf{uv}) = c + \mathbf{uv}$ but $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = (c + \mathbf{u}) \oplus (c + \mathbf{v}) = (c + \mathbf{u})(c + \mathbf{v})$ so these are not equal.

   Property 6: $(c + d) \odot \mathbf{u} = c + d + \mathbf{u}$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = (c + \mathbf{u})(d + \mathbf{u})$ which are not equal.

   Property 7: $c \odot (d \odot \mathbf{u}) = c + d + \mathbf{u}$ and $(cd) \odot \mathbf{u} = cd + \mathbf{u}$ which are not equal

   Property 8: $1 \odot \mathbf{u} = 1 + \mathbf{u} \neq \mathbf{u}$

   (b) $V = P_2$ with $p(t) \oplus q(t) = p'(t)q'(t)$ and $c \odot p(t) = cp(t)$.

   This is closed under $\oplus$ and $\odot$ and satisfied properties 1, 7, and 8 but fails properties 2-6.

   Closed under $\oplus$ and $\odot$: If $p(t)$ and $q(t)$ have degree at most 2, then their derivatives have degree at most 1 so the product of their derivatives has degree at most 2 and thus $p(t) \oplus q(t) = p'(t)q'(t)$ is also in P_2. Also, if $p(t)$ has degree at most 2, so does $c \odot p(t) = cp(t)$.

Property 1: $p(t) \oplus q(t) = p'(t)q'(t) = q'(t)p'(t) = q(t) \oplus p(t)$ so this is satisfied.

Property 2: $p(t) \oplus (q(t) \oplus r(t)) = p(t) \oplus q'(t)r'(t) = p'(t)(q'(t)r'(t))' = p'(t)(q'(t)r''(t) + q''(t)r'(t))$ and $(p(t) \oplus q(t)) \oplus r(t) = p'(t)q'(t) \oplus r(t) = (p'(t)q''(t) + p''(t)q'(t))r'(t)$ so these are not equal.

Property 3: There is no polynomial $e(t)$ such that $p(t) \oplus e(t) = p(t)$ for all $p(t)$. For example, take $p(t) = 1$ (the constant function). Then $p'(t) = 0$ so $p(t) \oplus e(t) = 0$ and is never equal to $p(t)$ no matter what we pick for $e(t)$. So there is no zero element and this condition fails.

Property 4: This condition automatically fails since there is no zero element.

Property 5: $c \odot (p(t) \oplus q(t)) = c \odot p'(t)q'(t) = cp'(t)q'(t)$ and $c \odot p(t) \oplus c \odot q(t) = cp(t) \oplus cq(t) = (cp(t))'(cq(t))' = c^2 p'(t)q'(t)$ so these are not equal.

Property 6: $(c+d) \odot p(t) = (c+d)p(t)$ and $c \odot p(t) \oplus d \odot p(t) = cp(t) \oplus dp(t) = cd(p'(t))^2$ so these are not equal.

Property 7: $c \odot (d \odot p(t)) = c \odot dp(t) = cdp(t) = (cd) \odot p(t)$ so this condition is satisfied.

Property 8: $1 \odot p(t) = p(t)$ so this condition is satisfied.

(c) V the set with two elements $\{\mathbf{v_1}, \mathbf{v_2}\}$ where $\mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1} = \mathbf{v_2}$ and $c \odot \mathbf{v_1} = c \odot \mathbf{v_2} = \mathbf{v_1}$.

This is closed under $\oplus$ and $\odot$ and satisfies properties 1-7 of the definition but fails property 8.

Closed under $\oplus$ and $\odot$: the sum of any two elements of the set is also an element of the set and the scalar multiple is always $\mathbf{v_1}$ which is in the set.

Property 1: $\oplus$ is defined to have $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1}$ so this holds.

Property 2: Clearly this holds if all three vectors are the same, so we need to consider the cases where the three vectors are not all the same. If you add two $\mathbf{v_1}$'s and one $\mathbf{v_2}$ you always get $\mathbf{v_2}$, for example $(\mathbf{v_1} \oplus \mathbf{v_1}) \oplus \mathbf{v_2} = \mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2}$ and $\mathbf{v_1} \oplus (\mathbf{v_1} \oplus \mathbf{v_2}) = \mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2}$. If you add one $\mathbf{v_1}$ and two $\mathbf{v_2}$'s you always get $\mathbf{v_1}$, for example $(\mathbf{v_1} \oplus \mathbf{v_2}) \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus (\mathbf{v_2} \oplus \mathbf{v_2}) = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$. This property always holds.

Property 3: The vector $\mathbf{v_1}$ is the zero vector.

Property 4: Each element is its own negative since when you add either one to itself you get $\mathbf{v_1}$ which is the zero vector.

Property 5: Scalar multiplication by c always gives you $\mathbf{v_1}$ so for any vectors $\mathbf{u}, \mathbf{v}$, $c \odot (\mathbf{u} \oplus \mathbf{v}) = \mathbf{v_1}$ and $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$ so these are equal.

Property 6: $(c+d) \odot \mathbf{u} = \mathbf{v_1}$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$ which are equal.

Property 7: $c \odot (d \odot \mathbf{u}) = c \odot \mathbf{v_1} = \mathbf{v_1}$ and $(cd) \odot \mathbf{u} = \mathbf{v_1}$ which are equal

Property 8: $1 \odot \mathbf{u} = \mathbf{v_1}$ which is not equal to $\mathbf{u}$ if $\mathbf{u} = \mathbf{v_2}$.

(d) V is matrices of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with operations $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ (matrix multiplication) and $r \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ra \\ 0 & 1 \end{bmatrix}$.

This is a vector space.

Closed under $\oplus$ and $\odot$: V is closed under $\oplus$ because $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$ which is in V. It is clear from the definition of $\odot$ that V is closed under $\odot$.

Property 1: Although this property does not hold for matrices in general, it will hold for the matrices which are in V because $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b+a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

Property 2: This property holds because $\oplus$ is defined to be matrix multiplication and in Chapter 1 we proved this property for matrix multiplication.

Property 3: Because $\oplus$ is matrix multiplication, for any A in V, we have $A \oplus I_2 = A$ so the zero vector will be $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is in V.

Property 4: As the zero vector is I_2 and $\oplus$ is matrix multiplication, the negative of a matrix will be its inverse. We need to check that the matrices in V are invertible and that their inverses are also in V. This holds because $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

Property 5: $c \odot \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) = c \odot \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & c(a+b) \\ 0 & 1 \end{bmatrix}$ and $c \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus c \odot \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ca \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & cb \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ca+cb \\ 0 & 1 \end{bmatrix}$ so these are equal.

Property 6: $(c+d) \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (c+d)a \\ 0 & 1 \end{bmatrix}$. Also $c \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus d \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ca \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & da \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ca+da \\ 0 & 1 \end{bmatrix}$ so these are equal.

Property 7: $c \odot \left(d \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) = c \odot \begin{bmatrix} 1 & da \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & cda \\ 0 & 1 \end{bmatrix} = (cd) \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

Property 8: $1 \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

2. Determine if W is a subspace of V. If it is, find a basis for W and $\dim W$.

(a) $V = \mathbb{R}^4$, let W be all 4-vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a = b + c$ and $d = a - b$.

W is a subspace of V. To prove that W is a subspace, we check that W is nonempty and closed under addition and scalar multiplication. W is nonempty as the zero vector is in W. To check closed under addition, take two vectors in V, say $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$ with $a_1 = b_1 + c_1, d_1 = a_1 - b_1$ and $\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$ with $a_2 = b_2 + c_2, d_2 = a_2 - b_2$, and check if their sum is still in V. Their sum is $\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix}$ and to check if it is in V we see if $(a_1 + a_2) = (b_1 + b_2) + (c_1 + c_2)$ and $(d_1 + d_2) = (a_1 + a_2) - (b_1 + b_2)$. As $a_1 = b_1 + c_1$ and $a_2 = b_2 + c + 2$, we get that $a_1 + a_2 = (b_1 + c_1) + (b_2 + c_2) = (b_1 + b_2) + (c_1 + c_2)$ so the first equation holds. Similarly, $d_1 = a_1 - b_1$ and $d_2 = a_2 - b_2$ so $d_1 + d_2 = (a_1 - b_1) + (a_2 - b_2) = (a_1 + a_2) - (b_1 + b_2)$. To check closed under scalar multiplication, take a vector in V, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a = b + c$ and $d = a - b$, and a real number r and check if the scalar multiple is still in V. The scalar multiple is $\begin{bmatrix} ra \\ rb \\ rc \\ rd \end{bmatrix}$ so it will be in V if $ra = rb + rc$ and $rd = ra - rb$. Both equations hold as they are just r times $a = b + c$ and $d = a - b$.

An alternate way to prove that W is a subspace is by recognizing W as something that we have already shown to be a subspace. W is actually the set of solutions to the homogeneous linear system $a - b - c = 0, a - b - d = 0,$

so it is the null space of the matrix $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$. As W is the null space of a 2×4 matrix, it is a subspace of $\mathbb{R}^4$. One way to find a basis for W is to use the method of finding bases for null spaces. The RREF of the matrix is $\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. There are no leading ones in columns 2 or 4, so b, d are anything and $a = b + d, c = d$. The null space is all matrices of the form $\begin{bmatrix} b+d \\ b \\ d \\ d \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. The dimension of W is 2.

Another way to prove W is a subspace and find a basis, is to write W as a span. If we plug in $b+c$ for a we get that W is matrices of the form $\begin{bmatrix} b+c \\ b \\ c \\ d \end{bmatrix}$ with $d = (b+c) - b = c$. Plugging in $d = c$ we get that W is vectors of the form $\begin{bmatrix} b+c \\ b \\ c \\ c \end{bmatrix}$. This is the same as vectors of the form $b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ so $W = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Spans are always subspaces, so this shows that W is a subspace. The two vectors are linearly independent, so they are a basis for W and again we get the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for W and that the dimension of W is 2.

(b) $V = \mathbb{R}^4$, let W be all 4-vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $ab = cd$

This is not a subspace. It is nonempty (it contains the zero vector) and closed under scalar multiplication, but it is not closed under addition. For example, the vectors $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ are both in V, but their sum is $\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$ which is not in V.

(c) $V = M_{22}$, let W be the set of matrices A such that $A\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ is a consistent linear system.

This is not a subspace. It is nonempty (it contains I_2). However W does not contain the zero matrix and is thus not a subspace of M_{22}. It is also not closed under addition or scalar multiplication. For example, I_2 and $-I_2$ are both in W but their sum is the zero matrix which is not in W so it is not closed under addition. It's also not closed under scalar multiplication because I_2 is in W but the scalar multiple $0I_2$ is not.

(d) $V = M_{22}$, let W be the set of matrices A such that $A\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}1&1\\1&1\end{bmatrix}A$.

This is a subspace. Denote the all ones matrix as $J = \begin{bmatrix}1&1\\1&1\end{bmatrix}$. The set W is nonempty as it contains the identity matrix and the zero matrix and J. If A_1, A_2 are in W then $A_1J = JA_1$ and $A_2J = JA_2$ so $(A_1 + A_2)J = A_1J + A_2J = JA_1 + JA_2 = J(A_1 + A_2)$ so $A_1 + A_2$ is also in W (this shows W is closed under addition). If A is in W and r is a real number then $(rA)J = rAJ = rJA = J(rA)$ so rA is also in W (this shows W is closed under scalar multiplication).

Suppose $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ is a 2×2 matrix. Then $JA = \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a+c&b+d\\a+c&b+d\end{bmatrix}$ and $AJ = \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}a+b&a+b\\c+d&c+d\end{bmatrix}$. The matrix A is in W if and only if $AJ = JA$ which happens if and only if $a = d$ and $b = c$. The set W is therefore all 2×2 matrices of the form $\begin{bmatrix}a&b\\b&a\end{bmatrix} = a\begin{bmatrix}1&0\\0&1\end{bmatrix} + b\begin{bmatrix}0&1\\1&0\end{bmatrix}$. The set $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ spans W and is linearly independent so it is a basis for W. The dimension of W is 2.

3. Let U and W be subspaces of a vector space V. The set of all vectors which are in both U and W is called the *intersection* of U and W and is denoted $U \cap W$.

(a) Prove that $U \cap W$ is a subspace of V.

$U \cap W$ is nonempty as $\mathbf{0}$ is in both U and W so it is in $U \cap W$. Suppose $\mathbf{v_1}$ and $\mathbf{v_2}$ are two vectors in $U \cap W$. Then $\mathbf{v_1}, \mathbf{v_2}$ are in U and U is a subspace so $\mathbf{v_1} + \mathbf{v_2}$ is also in U. The vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are also in W and W is a subspace so $\mathbf{v_1} + \mathbf{v_2}$ is in W. This shows that $\mathbf{v_1} + \mathbf{v_2}$ is in both U and W so it is in $U \cap W$. This proved that $U \cap W$ is closed under addition. Suppose $\mathbf{v}$ is in $U \cap W$ and r is a real number. Then $\mathbf{v}$ is in U and U is a subspace so $r\mathbf{v}$ is in U and $\mathbf{v}$ is in W which is a subspace so $r\mathbf{v}$ is in W. This shows that $r\mathbf{v}$ is in $U \cap W$ so $U \cap W$ is also closed under scalar multiplication.

(b) Let $V = \mathbb{R}^3$ and $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Find $U \cap W$.

Note that U and W are 2-dimensional subspaces of $\mathbb{R}^3$, so they are planes through the origin in $\mathbb{R}^3$. They are not the same plane, since the vectors in U all have that the first and second entries are equal, but the vectors in W do not all have this property. We see that U and W are 2 different planes whose intersection contains the origin, so the intersection of these two planes will be a line through the origin. This is a 1-dimensional subspace of $\mathbb{R}^3$.

Let $\mathbf{v}$ be a vector in $U \cap W$. The vectors in U look like $a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a \\ b \end{bmatrix}$, so $\mathbf{v} = \begin{bmatrix} a \\ a \\ b \end{bmatrix}$ for some real numbers a, b. The vectors in W look like $c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \\ c \end{bmatrix}$, so $\mathbf{v} = \begin{bmatrix} c \\ d \\ c \end{bmatrix}$ for some real numbers c, d. We thus have that $\begin{bmatrix} a \\ a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \\ c \end{bmatrix}$ so $a = c, a = d, b = c$. We see that we must have $a = b = c = d$ with a any real number. Therefore $\mathbf{v} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ with a any real number. The vectors in $U \cap W$ are exactly the scalar multiples of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $U \cap W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a 1-dimensional subspace of $\mathbb{R}^3$ as expected).

4. Let U and W be subspaces of a finite dimensional vector space V. Let $U+W$ be the set of all vectors in V that have the form $\mathbf{u}+\mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W.

   (a) Show that $U+W$ is a subspace of V.

   $U+W$ is nonempty as $\mathbf{0}$ is in U and $\mathbf{0}$ is in W so $\mathbf{0}+\mathbf{0}=\mathbf{0}$ is in $U+W$. We therefore need to check that it is closed under addition and scalar multiplication. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be any two vectors in $U+W$. As $\mathbf{v_1}$ and $\mathbf{v_2}$ are in $U+W$, we can write them as $\mathbf{v_1}=\mathbf{u_1}+\mathbf{w_1}$ and $\mathbf{v_2}=\mathbf{u_2}+\mathbf{w_2}$ where $\mathbf{u_1}$ and $\mathbf{u_2}$ are in U and $\mathbf{w_1}$ and $\mathbf{w_2}$ are in W. The sum is $\mathbf{v_1}+\mathbf{v_2}=\mathbf{u_1}+\mathbf{w_1}+\mathbf{u_2}+\mathbf{w_2}=(\mathbf{u_1}+\mathbf{u_2})+(\mathbf{w_1}+\mathbf{w_2})$. U and W are subspaces so $\mathbf{u_1}+\mathbf{u_2}$ is in U and $\mathbf{w_1}+\mathbf{w_2}$ is in W and hence we have written $\mathbf{v_1}+\mathbf{v_2}$ as a sum of a vector in U and a vector in W so $\mathbf{v_1}+\mathbf{v_2}$ is in $U+W$. This shows that $U+W$ is closed under addition. Let $\mathbf{v}$ be any vector in $U+W$ and r be any scalar. Then $\mathbf{v}=\mathbf{u}+\mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W so $r\mathbf{v}=r\mathbf{u}+r\mathbf{w}$. As U and W are subspaces, $r\mathbf{u}$ is in U and $r\mathbf{w}$ is in W so $r\mathbf{v}$ is the sum of a vector in U and a vector in W so it is in $U+W$. This shows $U+W$ is closed under scalar multiplication.

   (b) Show that $\dim U+W \leq \dim U + \dim W$.

   U and V are subspaces of a finite dimensional vector space V, so they are finite dimensional. Suppose $\dim U = n$ and $\dim W = m$. Let $S=\{\mathbf{u_1},\mathbf{u_2},...,\mathbf{u_n}\}$ be a basis for U and let $T=\{\mathbf{w_1},\mathbf{w_2},...,\mathbf{w_m}\}$ be a basis for W. Take R to be the set $R=\{\mathbf{u_1},\mathbf{u_2},...,\mathbf{u_n},\mathbf{w_1},\mathbf{w_2},...,\mathbf{w_m}\}$. Note that $U+W$ contains both U and W so all the vectors in R are in $U+W$. We will show that R is a spanning set for $U+W$. Any vector in $U+W$ can be written in the form $\mathbf{u}+\mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W. As S is a basis for U, we can write $\mathbf{u}=a_1\mathbf{u_1}+...+a_n\mathbf{u_n}$ for some real numbers a_i and similarly we can write $\mathbf{w}=b_1\mathbf{w_1}+...+b_m\mathbf{w_m}$ for some real numbers b_i. Then $\mathbf{u}+\mathbf{w}=\mathbf{u}=a_1\mathbf{u_1}+...+a_n\mathbf{u_n}+b_1\mathbf{w_1}+...+b_m\mathbf{w_m}$ so any vector in $U+W$ is a linear combination of the vectors in R. It follows that R is a spanning set for $U+W$ so R contains a basis for $U+W$. Thus any basis for $U+W$ has size less than or equal to the size of R (which is $n+m$) so $\dim U+W \leq n+m = \dim U + \dim W$.

   Note: The above proof assumed that U and W both had bases. If one (or both) of them is the zero vectors space, then it will not have a basis. The result still holds, and in that case $\dim U+W = \dim U + \dim W$. For example, if U is the zero vector space, then $U+W=W$ and $\dim U = 0$ so $\dim U+W = \dim W = \dim W + 0 = \dim W + \dim U$.

   (c) **Challenge: Prove $\dim U+W = \dim U + \dim W - \dim U \cap W$.

(this is more difficult than an exam problem)

See separate file.

5. For what value or values of c is the set $\left\{ \begin{bmatrix} 3 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} c \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ c \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?.

   Hint: Use determinants.

   Given n vectors in $\mathbb{R}^n$, we can check if they are a basis for $\mathbb{R}^n$ by forming the $n \times n$ matrix A whose columns are the n vectors and seeing if the rank of A is n. One way to check this is to take the determinant of A. If it is nonzero, then the rank is n and the column vectors are a basis for $\mathbb{R}^n$. In this case, $A = \begin{bmatrix} 3 & c & 1 \\ -5 & 2 & c \\ -4 & 0 & c \end{bmatrix}$. $\det(A) = 6c + (-4c^2) + 0 - (-8) - (0) - (-5c^2) = c^2 + 6c + 8 = (c+2)(c+4)$. This is 0 when $c = -2, -4$ and nonzero otherwise. Therefore the set is a basis for $\mathbb{R}^3$ when $c \neq -2, -4$.

6. For each set S, determine if S contains a basis for $\mathbb{R}^3$, is contained in a basis for $\mathbb{R}^3$, both, or neither.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

   Contained in a basis. It is linearly independent so it must be contained in a basis and it is too small to contain a basis.

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

   Contains a basis. This set spans $\mathbb{R}^3$ so it contains a basis but it is too big to be contained in a basis.

   (c) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$

   Both. This set is both linearly independent and spans $\mathbb{R}^3$ so it is a basis and therefore both contains and is contained in a basis.

(d) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

Neither. This set is not linearly independent (the last vector is the second minus the first) and its span has dimension 2 so is not all of $\mathbb{R}^3$. Since it is not linearly independent it cannot be contained in a basis and it does not span so it cannot contain a basis.

7. Find a basis for span S where S is the following subset of P_3.

$$S = \{t^3 + t^2 - 1, t^2 + 2t + 1, t^3 + 2t^2 + 2t, t^3 + t - 1, -t^3 - 5t^2 + t\}$$

We take a linear combination and set it equal to $\mathbf{0}$ as follows: $a(t^3 + t^2 - 1) + b(t^2 + 2t + 1) + c(t^3 + 2t^2 + 2t) + d(t^3 + t - 1) + e(-t^3 - 5t^2 + t) = 0$. This is the same as $(a+c+d-e)t^3 + (a+b+2c-5e)t^2 + (2b+2c+d+e)t + (-a+b-d) = 0$ which gives us the linear system $a+c+d-e = 0, a+b+2c-5e = 0, 2b+2c+d+e = 0, -a+b-d = 0$. This has coefficient matrix $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & -5 \\ 0 & 2 & 2 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{bmatrix}$. Doing row operations gets REF of $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The leading ones are in columns 1,2, and 4 so we can take the basis to be the first, second, and fourth elements of S. The basis we get is $\{t^3 + t^2 - 1, t^2 + 2t + 1, t^3 + t - 1, \}$.

8. Let $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$. W is a subspace of $\mathbb{R}^3$.

(a) What is the dimension of W?

W is spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$ and this set is also linearly independent, so it is a basis for W and therefore W has dimension 2.

(b) Is $W = \mathbb{R}^3$? If not, find a vector in $\mathbb{R}^3$ which is not in W.

W has dimension 2 so it is not equal to $\mathbb{R}^3$. W is a plane in $\mathbb{R}^3$ so there are lots of vectors in $\mathbb{R}^3$ which are not in W. One example would be $\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$.

To show that this vector is not in W, take $a\begin{bmatrix}1\\1\\2\end{bmatrix}+b\begin{bmatrix}-1\\4\\2\end{bmatrix}=\begin{bmatrix}0\\5\\0\end{bmatrix}$. Then $a-b=0, a+4b=5, 2a+2b=2$, but the first two equations give $a=b=1$ which does not work in $2a+2b=2$, so this system has no solutions and the vector is not in W.

(c) Find a basis for $\mathbb{R}^3$ which contains $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\4\\2\end{bmatrix}$.

Bases for $\mathbb{R}^3$ have size 3 and we already know two of the vectors that must be in the basis. So we just need to find one more vector to include. Any vector which is not a linear combination of $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\4\\2\end{bmatrix}$ will work. In particular, we can use our answer to the previous part. So one possible basis would be $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}-1\\4\\2\end{bmatrix},\begin{bmatrix}0\\5\\0\end{bmatrix}\right\}$.

(d) Prove that the vector $\begin{bmatrix}5\\0\\6\end{bmatrix}$ is in W.

To prove it is in W, we need to write it as a linear combination of the two vectors that span W. If $a\begin{bmatrix}1\\1\\2\end{bmatrix}+b\begin{bmatrix}-1\\4\\2\end{bmatrix}=\begin{bmatrix}5\\0\\6\end{bmatrix}$ then $a-b=5, a+4b=0, 2a+2b=6$. Solving this linear system, we get that $a=4, b=-1$ so $\begin{bmatrix}5\\0\\6\end{bmatrix}=4\begin{bmatrix}1\\1\\2\end{bmatrix}-\begin{bmatrix}-1\\4\\2\end{bmatrix}$ and the vector is in W.

(e) Find a basis for W which contains the vector $\begin{bmatrix}5\\0\\6\end{bmatrix}$.

W is a 2-dimensional space, so any basis for W will have size 2. The basis needs to contain $\begin{bmatrix}5\\0\\6\end{bmatrix}$ so we just need to find one other vector to include in our basis. Any vector which is in W and is not a multiple of $\begin{bmatrix}5\\0\\6\end{bmatrix}$ will work. For example, one possible basis would be $\left\{ \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Another possible basis would be $\left\{ \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$.

9. Determine if the statement is true or false. If it is true, give a proof. If it is false, find a counterexample.

   (a) If V has basis S and W is a subspace of V, then there exists a set T contained in S which is a basis for W.

   False. For example take $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\}$. Then S is a basis for V and W is a subspace of V. Note that W is 1-dimensional so any basis for W consists of exactly one vector. However S cannot contain a basis for W since the vectors in S are not in W.

   (b) If W is a subspace of V and both W and V are infinite dimensional, then $W = V$.

   False. For example take V to be P (the space of all polynomials) and take W to be the set of all polynomials with constant term 0. W is a subspace of V and they are both infinite dimensional, but they are not equal as V contains things like $t + 1$ which are not in W.

   (c) If V is a subspace of $\mathbb{R}^3$ and V contains the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, then V also contains the vector $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

   True. The vector $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. In particular, $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. V is a subspace so it is closed under addition and scalar multiplication and therefore if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ are in V, then so are all the linear combinations of these vectors including $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

(d) If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a set of linearly independent vectors in a vector space V and $\mathbf{w}$ is a nonzero vector in V then the set $\{\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, ..., \mathbf{v_k} + \mathbf{w}\}$ is also linearly independent.

False. For example, we could take $\mathbf{w} = -\mathbf{v_1}$. This is nonzero since $\mathbf{v_1}$ is in S which is linearly independent, but the new set will contain $\mathbf{0}$ so it will be linearly dependent.

10. Suppose A and B are $m \times n$ matrices and that the RREF of A and B are the same. Which of the following must be the same for A and B: the rank, the nullity, the row space, the column space, the null space?

The rank, nullity, row space, and null space will be the same but the column space may be different. The rank and nullity can be determined from the RREF by counting columns with and without leading ones, so these will be the same. Row operations do not change the row space, so the row spaces of A and B are both equal to the row spaces of their RREF which is the same. The null spaces are the solutions to $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ which are determined by the RREF so these will be the same. Row operations may change the column space though, so the column space doesn't have to be the same. For example, the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ both have RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ but their column spaces are span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ which are not the same.

11. Let A be an $n \times n$ matrix. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be a basis for $\mathbb{R}^n$ and let $T = \{A\mathbf{v_1}, A\mathbf{v_2}, ..., A\mathbf{v_n}\}$.

(a) Prove that if A is invertible, then T is linearly independent.

To show that T is linearly independent, take a linear combination of the vectors in T and set it equal to $\mathbf{0}$. This gives us the equation $a_1 A\mathbf{v_1} + a_2 A\mathbf{v_2} + ... + a_n A\mathbf{v_n} = \mathbf{0}$. This can be rewritten as $A(a_1\mathbf{v_1}) + A(a_2\mathbf{v_2}) + ... + A(a_n\mathbf{v_n}) = \mathbf{0}$ or $A(a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_n\mathbf{v_n}) = \mathbf{0}$. We need to prove that all the a_i must equal 0. As A is invertible, we can multiply both sides of this equation by A^{-1} on the left to get $A^{-1}A(a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_n\mathbf{v_n}) = A^{-1}\mathbf{0}$ which simplifies to $a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_n\mathbf{v_n} = \mathbf{0}$. By the linear independence of S, all a_i must equal 0.

(b) Prove that for any $\mathbf{v}$ in $\mathbb{R}^n$, the n-vector $A\mathbf{v}$ is in the column space of A.

Let $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and denote the columns of A as $\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}$. Then $A\mathbf{v} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{c_1} + x_2\mathbf{c_2} + ... + x_n\mathbf{c_n}$. We see that $A\mathbf{v}$ is a linear combination of the columns of A so it is in the column space of A.

(c) Prove that if the rank of A is less than n, then T does not span $\mathbb{R}^n$.

By part (b), all the vectors in T are contained in the column space of A and hence span T is contained in the column space of A. If the rank of A is less than n, then the dimension of the column space of A is less than n. As span T is contained in the column space of A, it also has dimension less than n so span T cannot be all of $\mathbb{R}^n$.

(d) Use the previous parts to show that T is a basis for $\mathbb{R}^n$ if and only if A has rank n.

If A has rank n, then A is invertible. By part (a), T is linearly independent and it consists of n vectors in $\mathbb{R}^n$ so T must be a basis for $\mathbb{R}^n$. If A does not have rank n, then it must have rank less than n. By part (b), T would not span so T would not be a basis.

12. Let A be a 3×6 matrix.

(a) What are the possible values for the rank of A?

The rank can be 0, 1, 2, 3. It cannot be any larger because the dimension of the row space cannot be larger than 3 since there are three rows.

(b) What can you say about the nullity of A?

The nullity is equal to the number of columns, which is 6, minus the rank. The possible values for rank are 0, 1, 2, 3 so the possible values for the nullity are 6, 5, 4, 3.

(c) Suppose that the rank of A is 3. Are the rows of A linearly independent? Are the columns of A linearly independent?

If the rank is 3 then both the row and column spaces have dimension 3. There are 3 rows and their span (the row space) is dimension 3 so they must be linearly independent. There are 6 columns and the dimension of their span is 3 so they are not linearly independent.

13. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ -2 & -4 & 1 & 1 \end{bmatrix}$.

(a) Find the rank and nullity of A.
The first four parts of this problem can be done by finding REF or RREF of A. Doing the row operations $r_3 + 2r_1 \to r_3, r_3 - r_2 \to r_3$ gives us the REF of A which is $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. In this case it is also the RREF of A.
There are two columns containing leading ones (1 and 3) so the rank is 2. There are two columns which do not contain leading ones (2 and 4) so the nullity is 2.

(b) Find a basis for the row space of A.

The nonzero rows in REF of A are a basis for the row space of A so a basis is $\left\{ \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix} \right\}$.

(c) Find a basis for the column space of A.

The column vectors of A corresponding to the columns of REF with leading ones (1 and 3) are a basis for the column space so the first and third columns of A are a basis for the column space of A. The basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) Find a basis for the null space of A.

The null space is the set of solutions to $A\mathbf{x} = \mathbf{0}$. If we use the variables x, y, z, w then we see from the RREF of A that columns 2 and 4 do not contain leading ones so the variables y, w can be any real numbers and $z = -3w, x = -2y - w$ so the null space is all vectors of the form $\begin{bmatrix} -2y - w \\ y \\ -3w \\ w \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$. A basis for this space is

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-3\\1\end{bmatrix}\right\}.$$

(e) Let $\mathbf{b} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. Prove that $\begin{bmatrix}-1\\1\\2\\0\end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and find all the other solutions to $A\mathbf{x} = \mathbf{b}$.

$A\begin{bmatrix}-1\\1\\2\\0\end{bmatrix} = \begin{bmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 3\\-2 & -4 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} = \mathbf{b}$, so it is a solution to $A\mathbf{x} = \mathbf{b}$. The solutions to $A\mathbf{x} = \mathbf{b}$ will be things of the form $\begin{bmatrix}-1\\1\\2\\0\end{bmatrix} + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{0}$. Therefore the solutions are all vectors of the form $\begin{bmatrix}-1\\1\\2\\0\end{bmatrix} + \begin{bmatrix}-2y-w\\y\\-3w\\w\end{bmatrix} = \begin{bmatrix}-1-2y-w\\1+y\\2-3w\\w\end{bmatrix}$.