Review for Exam 2

Note: All vector spaces are real vector spaces. Definition 4.4 will be provided on the exam as it appears in the textbook.

1. Determine if the set V together with operations $\oplus$ and $\odot$ is a vector space. Either show that Definition 4.4 is satisfied or determine which properties of Definition 4.4 fail to hold.

   (a) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = c + \mathbf{u}$.

   (b) $V = P_2$ with $p(t) \oplus q(t) = p'(t)q'(t)$ and $c \odot p(t) = cp(t)$.

   (c) V the set with two elements $\{\mathbf{v_1}, \mathbf{v_2}\}$ where $\mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1} = \mathbf{v_2}$ and $c \odot \mathbf{v_1} = c \odot \mathbf{v_2} = \mathbf{v_1}$.

   (d) V is matrices of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with operations $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ (matrix multiplication) and $r \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ra \\ 0 & 1 \end{bmatrix}$.

2. Determine if W is a subspace of V. If it is, find a basis for W and $\dim W$.

   (a) $V = \mathbb{R}^4$, let W be all 4-vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a = b + c$ and $d = a - b$.

   (b) $V = \mathbb{R}^4$, let W be all 4-vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $ab = cd$

   (c) $V = M_{22}$, let W be the set of matrices A such that $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a consistent linear system.

   (d) $V = M_{22}$, let W be the set of matrices A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.

3. Let U and W be subspaces of a vector space V. The set of all vectors which are in both U and W is called the *intersection* of U and W and is denoted $U \cap W$.

   (a) Prove that $U \cap W$ is a subspace of V.

   (b) Let $V = \mathbb{R}^3$ and $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   Find $U \cap W$.

4. Let U and W be subspaces of a finite dimensional vector space V. Let $U + W$ be the set of all vectors in V that have the form $\mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W.

   (a) Show that $U + W$ is a subspace of V.

   (b) Show that $\dim U + W \leq \dim U + \dim W$.

   (c) **Challenge: Prove $\dim U + W = \dim U + \dim W - \dim U \cap W$.
   (this is more difficult than an exam problem)

5. For what value or values of c is the set $\left\{ \begin{bmatrix} 3 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} c \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ c \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?.

   Hint: Use determinants.

6. For each set S, determine if S contains a basis for $\mathbb{R}^3$, is contained in a basis for $\mathbb{R}^3$, both, or neither.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

   (c) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$

   (d) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

7. Find a basis for span S where S is the following subset of P_3.

$$S = \{t^3 + t^2 - 1, t^2 + 2t + 1, t^3 + 2t^2 + 2t, t^3 + t - 1, -t^3 - 5t^2 + t\}$$

8. Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$. W is a subspace of $\mathbb{R}^3$.

   (a) What is the dimension of W?

   (b) Is $W = \mathbb{R}^3$? If not, find a vector in $\mathbb{R}^3$ which is not in W.

   (c) Find a basis for $\mathbb{R}^3$ which contains $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$.

   (d) Prove that the vector $\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$ is in W.

   (e) Find a basis for W which contains the vector $\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$.

9. Determine if the statement is true or false. If it is true, give a proof. If it is false, find a counterexample.

   (a) If V has basis S and W is a subspace of V, then there exists a set T contained in S which is a basis for W.

   (b) If W is a subspace of V and both W and V are infinite dimensional, then $W = V$.

   (c) If V is a subspace of $\mathbb{R}^3$ and V contains the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, then V also contains the vector $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

   (d) If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a set of linearly independent vectors in a vector space V and $\mathbf{w}$ is a nonzero vector in V then the set $\{\mathbf{v_1}+\mathbf{w}, \mathbf{v_2}+\mathbf{w}, ..., \mathbf{v_k}+\mathbf{w}\}$ is also linearly independent.

10. Suppose A and B are $m \times n$ matrices and that the RREF of A and B are the same. Which of the following must be the same for A and B: the rank, the nullity, the row space, the column space, the null space?

11. Let A be an $n \times n$ matrix. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be a basis for $\mathbb{R}^n$ and let $T = \{A\mathbf{v_1}, A\mathbf{v_2}, ..., A\mathbf{v_n}\}$.

(a) Prove that if A is invertible, then T is linearly independent.

(b) Prove that for any $\mathbf{v}$ in $\mathbb{R}^n$, the n-vector $A\mathbf{v}$ is in the column space of A.

(c) Prove that if the rank of A is less than n, then T does not span $\mathbb{R}^n$.

(d) Use the previous parts to show that T is a basis for $\mathbb{R}^n$ if and only if A has rank n.

12. Let A be a 3×6 matrix.

(a) What are the possible values for the rank of A?

(b) What can you say about the nullity of A?

(c) Suppose that the rank of A is 3. Are the rows of A linearly independent? Are the columns of A linearly independent?

13. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ -2 & -4 & 1 & 1 \end{bmatrix}$.

(a) Find the rank and nullity of A.

(b) Find a basis for the row space of A.

(c) Find a basis for the column space of A.

(d) Find a basis for the null space of A.

(e) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Prove that $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and find all the other solutions to $A\mathbf{x} = \mathbf{b}$.