Review for Exam 1

1. Find all a for which the following linear system has no solutions, one solution, and infinitely many solutions.

   $x + y - z = 2$
   $x + 2y + z = 3$
   $x + y + (a^2 - 5)z = a$

   The augmented matrix for this system is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right]$. Doing the elementary row operations $r_2 - r_1 \to r_2$ and $r_3 - r_1 \to r_3$, this becomes $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$. To get the matrix in row echelon form, we want to divide row three by $a^2 - 4$, but we can only do this if $a^2 - 4 \neq 0$ so we need to consider the cases where $a^2 - 4 = 0$ and $a^2 - 4 \neq 0$ separately. The case where $a^2 - 4 = 0$ splits into two cases, $a = 2$ and $a = -2$ so we have three cases to consider.

   If $a^2 - 4 \neq 0$, then the row echelon form of the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{a-2}{a^2-4} \end{array}\right]$. This gives us the equations $z = \frac{a-2}{a^2-4}$, $y + 2z = 1$ and $x + y - z = 2$. There are leading ones in each column corresponding to the three variables an no equations that look like $0 = 1$, so there is one solution. In particular, it is $z = \frac{a-2}{a^2-4}, y = 1 - \frac{2(a-2)}{a^2-4}, x = 2 - (1 - \frac{2(a-2)}{a^2-4}) + \frac{a-2}{a^2-4}$.

   If $a = 2$, then the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so the equations are $x + y - z = 2, y + 2z = 1, 0 = 0$. There are infinitely many solutions as z can be anything. In particular, the solutions are all vectors of the form $\begin{bmatrix} 1+3z \\ 1-2z \\ z \end{bmatrix}$ where z is anything.

   If $a = -2$ then the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$ and the last equation is $0 = -4$ which has no solutions.

Combining these results we get that the system has no solutions when $a = -2$, one solution when $a \neq 2, -2$, and infinitely many solutions when $a = 2$.

2. Find the augmented matrix of each system of linear equations. Use Gaussian elimination or Gauss-Jordan reduction to solve the linear system.

   (a) $y + 3z = -10$
   $x + 2z = 11$
   $2x - y + 7z = 14$

   The augmented matrix is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & -10 \\ 1 & 0 & 2 & 11 \\ 2 & -1 & 7 & 14 \end{array}\right]$. This can be gotten into row echelon form by doing the following row operations: $r_1 \leftrightarrow r_2$, $-2r_1 + r_3 \to r_3$, $r_3 + r_2 \to r_3$, $\frac{1}{6}r_3 \to r_3$. The resulting matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 11 \\ 0 & 1 & 3 & -10 \\ 0 & 0 & 1 & -3 \end{array}\right]$.
   We then proceed one of two different ways. One way to finish solving the problem is to write this as the equations $x+2z = 11, y+3z = -10, z = -3$. Using back substitution we get that $y = -10 + 9 = -1, x = 11 + 6 = 17$ so the solution is $\begin{bmatrix} 17 \\ -1 \\ -3 \end{bmatrix}$.
   Another way to finish the problem is to do row operations to get the matrix in reduced row echelon form. The following row operations will get the matrix into reduced row echelon form: $-3r_3 + r_2 \to r_2$, $-2r_3 + r_1 \to r_1$ and the resulting matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$. The equations are $x = 17, y = -1, z = -3$ so the solution is $\begin{bmatrix} 17 \\ -1 \\ -3 \end{bmatrix}$.

   (b) $x + 3y - z + w = 5$
   $x - 6y + 2z = 1$
   $2x + w = 6$

   The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 1 & -6 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 6 \end{array}\right]$. The following elementary row operations will get a matrix in row echelon form: $-r_1 + r_2 \to r_2$, $-2r_1 + r_3 \to r_3$, $-\frac{1}{9}r_2 \to r_2$, $6r_2 + r_3 \to r_3$, $-3r_3 \to r_3$. The resulting matrix is $\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 0 & 1 & -1/3 & 1/9 & 4/9 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$.

We see from here that $w = 4$. There is no leading one in the z column so z can be anything. To find y, use that $y - \frac{1}{3}z + \frac{1}{9}w = \frac{4}{9}$ so $y = \frac{1}{3}z$. Then $x + 3y - z + w = 5$ so $x - 3y + 3y + 4 = 5$ and $x = 1$. There are infinitely many solutions, they are all vectors of the form $\begin{bmatrix} 1 \\ \frac{1}{3}z \\ z \\ 4 \end{bmatrix}$ where z is anything.

Alternatively, we could do the row operations $-\frac{1}{9}r_3 + r_2 \to r_2, -r_3 + r_1 \to r_1, -3r_2 + r_1 \to r_1$ to get the matrix in reduced row echelon form. The resulting matrix is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$. We again see that $x = 1, w = 4$, and $y = \frac{1}{3}z$. So the solutions are vectors of the form $\begin{bmatrix} 1 \\ \frac{1}{3}z \\ z \\ 4 \end{bmatrix}$ where z is anything.

This could also be written as all vectors of the form $\begin{bmatrix} 1 \\ y \\ 3y \\ 4 \end{bmatrix}$ where y is anything.

(c) $2x + 3y + z - w = 1$
$x - y + w = 2$
$4x + y + z + w = 4$
$6x + 3y - 7z - w = 12$

The augmented matrix is $\left[\begin{array}{cccc|c} 2 & 3 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 2 \\ 4 & 1 & 1 & 1 & 4 \\ 6 & 3 & -7 & -1 & 12 \end{array}\right]$. The row operations $r_1 \leftrightarrow r_2, -2r_1 + r_2 \to r_2, -4r_1 + r_3 \to r_3, -6r_1 + r_4 \to r_4$ give us the matrix $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 5 & 1 & -3 & -3 \\ 0 & 5 & 1 & -3 & -4 \\ 0 & 9 & -7 & -7 & 0 \end{array}\right]$. We can see looking at rows 2 and 3 that there will not be any solutions because we cannot have $5y + z - 3w$ equal to both -3 and -4. We can also see this by taking $r_2 - r_3 \to r_2$. Then r_2 would be $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \end{array}\right]$ and there are no solutions to the equation $0 = 1$.

3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.
Compute $D = AB^T + 2C^2$. Which of the following terms describe D: diagonal, scalar, upper triangular, lower triangular, symmetric, skew symmetric, invertible.
Circle all (if any) that apply.

$AB^T = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$ and $C^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $D = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$.

This matrix is symmetric and invertible. It is not diagonal, scalar, upper triangular, lower triangular, or skew symmetric.

4. Let A be an $m \times n$ matrix with $n > m$ (so A has more columns than rows).

   (a) Prove that the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has infinite solutions.

   The system is homogeneous so it has at least one solution. Whether it has 1 or infinite solutions will depend on if the columns of the RREF of A all contain leading ones. Let B be the RREF of A. The augmented matrix of the linear system is $[A : \mathbf{0}]$ and the RREF of this matrix is $[B : \mathbf{0}]$. B is the same size as A so it has more columns then rows. There are at most m leading 1's (one per nonzero row) in B and there are n columns with $n > m$, so at least one columm of B does not contain a leading one. The variable corresponding to the column without a leading one is a free variable that can be anything so there will be infinite solutions.

   (b) What are the possible numbers of solutions to $A\mathbf{x} = \mathbf{b}$?

   By the same argument as above, we know that the RREF of A has a least one column without a leading one, so we can rule out that the possibility of having one solution. The other two possibilities are 0 or infinite. We will write down examples to show that both are possible. If A has more columns than rows, the linear system has more variables than equations. An example with no solutions would be $x + y + z = 1, x + y + z = 2$. An example with infinite solutions would be $x + y = 1, z = 1$. Therefore $A\mathbf{x} = \mathbf{b}$ can have 0 or infinitely many solutions.

5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Determine if A is a linear combination of the matrices B, C, D where $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

We want to see if we can find real numbers b, c, d such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix}$. This gives us the system $b+c-d = 1, b = 2, c = 3, d = 4$ which has solution $b = 2, c = 3, d = 4$ so

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore A is a linear combination of B, C, and D.

6. Let $A = \begin{bmatrix} 1 & 2 & y & z & 0 \\ 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & 0 & y+z \end{bmatrix}$.

Find all possible choices for the variables x, y, z for which A is in RREF.

To be in RREF, the first nonzero entry of each nonzero row must be 1. This forces x to be 0 or 1 and $y + z$ to be 0 or 1. If $x = 1$, then it is a leading one and the other entries in column 2 must be 0, so $y = 0$. The 1,4 entry can be anything so this does not put any restrictions on z, but as $y + z$ is 0 or 1 and $y = 0$ we see that z has to be 0 or 1. We have so far found two possibilities which are $x = 1, y = 0, z = 0$ or $x = 1, y = 0, z = 1$. If $x = 0$, then the entry above it can be anything. But then the 1 in the 2,4 position is a leading one so the other entries in column 4 must be 0 so $z = 0$. Then as $y + z$ is 0 or 1 and $z = 0$ we get that $y = 1$ or $y = 0$. This gives two more possibilities $x = 0, y = 0, z = 0$ or $x = 0, y = 1, z = 0$.

There are a total of 4 different wants to pick x, y, z.

i) $x = 1, y = 0, z = 0$

ii) $x = 1, y = 0, z = 1$

iii) $x = 0, y = 0, z = 0$

iv) $x = 0, y = 1, z = 0$

7. (a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$. Compute AB.

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

(b) Let C be an $n \times n$ diagonal matrix with diagonal entries $c_1, c_2, ..., c_n$ and D be an $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, ..., d_n$. Describe the matrix CD.

CD is also $n \times n$ and diagonal. The diagonal entries are $c_1 d_1, c_2 d_2, ..., c_n d_n$.

(c) Determine if the following statement is true or false.
If C and D are diagonal $n \times n$ matrices then $CD = DC$.

True. Use part (b). Both CD and DC are diagonal matrices. The entries on the diagonal of CD are $c_1 d_1, c_2 d_2, ..., c_n d_n$ and the entries on the diagonal of DC are $d_1 c_1, d_2 c_2, ..., d_n c_n$. The c_i and d_i are just numbers so $c_i d_i = d_i c_i$ for all i and the entries of CD and DC are all equal.

8. Determine if each statement is true or false. If it is true give a proof. If it is false find a counterexample.

(a) If $\mathbf{v}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$, then $5\mathbf{v}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

False. If $\mathbf{v}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{v} = \mathbf{b}$. Then $A(5\mathbf{v}) = 5(A\mathbf{v}) = 5\mathbf{b}$. This is not equal to $\mathbf{b}$ unless $\mathbf{b} = \mathbf{0}$, so this fails for any consistent linear system which is not homogeneous.

(b) If A is an $n \times n$ matrix and $A^k = I_n$ for some positive integer k, then A is invertible.

TRUE. Suppose $A^k = I_n$. If $k = 1$, then $A = I_n$ so A is invertible. If $k > 1$, then we can rewrite A^k as $A^{k-1}A$ or AA^{k-1} so we see that $AA^{k-1} = A^{k-1}A = I_n$ so A has inverse A^{k-1}.

Another way to prove this is using determinants. If $A^k = I_n$, then $\det(A)^k = 1$ so $\det(A) = -1, 1$. As $\det(A) \neq 0$, A is invertible.

(c) If A is an invertible $n \times n$ matrix, then $A^k = I_n$ for some positive integer k.

False. For example, the matrix $2I$ is invertible (it has inverse $\frac{1}{2}I$), but $(2I)^k = 2^k I$ which does not equal I for any positive integer k.

(d) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(A^2 - A) = 6$.

False. There is no nice property for the determinant of a difference of matrices, so there is no way to find $\det(A^2 - A)$ from just $\det(A)$. For example if $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, then $\det(A) = 3$ but $A^2 - A = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$ which has determinant 0, not 6.

9. Find the inverse of A or show that A is not invertible.

(a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 3 & 4 & 5 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{array}\right]$. Perform the following row operations: $-2r_1+r_2 \to r_2$, $-3r_1+r_3 \to r_3$, $r_2 \leftrightarrow r_3$, $\frac{1}{4}r_2 \to r_2$, $-r_3 \to r_3$, $\frac{1}{4}r_3 + r_2 \to r_2$, and $-2r_3 + r_1 \to r_1$.

The resulting matrix is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$, so the inverse of is $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -1/4 & -1/4 & 1/4 \\ 2 & -1 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 7 & 5 \\ 3 & -1 & 2 \\ 5 & 13 & 12 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{ccc|ccc} 1 & 7 & 5 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 5 & 13 & 12 & 0 & 0 & 1 \end{array}\right]$. If we do the row operations $-3r_1 + r_2 \to r_2, -5r_1 + r_3 \to r_3$, and $-r_2 + r_3 \to r_3$ we get $\left[\begin{array}{ccc|ccc} 1 & 7 & 5 & 1 & 0 & 0 \\ 0 & -22 & -13 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$. We can stop here because A is row equivalent to matrix with a row of zeros so the RREF of A is not going to be I_3 and A is not invertible.

(c) $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$. Perform the following row operations: $-2r_1 + r_4 \to r_4$, $r_2 \leftrightarrow r_3$, $-3r_2 + r_3 \to r_3$, $r_3 \leftrightarrow r_4$, $\frac{1}{2}r_3 \to r_3$, $-\frac{1}{12}r_4 \to r_4$, $-\frac{1}{2}r_4 + r_3 \to r_3$, $-4r_4 + r_2 \to r_2$, and $r_3 + r_1 \to r_1$.

The resulting matrix is $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1/24 & -1/8 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1/24 & -1/8 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & -1/12 & 1/4 & 0 \end{array}\right]$, so the inverse of is $A^{-1} = \frac{1}{24}\begin{bmatrix} 0 & 1 & -3 & 12 \\ 0 & 8 & 0 & 0 \\ -24 & 1 & -3 & 12 \\ 0 & -2 & 6 & 0 \end{bmatrix}$.

10. Let A be an $n \times n$ matrix such that the n-th row is a linear combination of rows 1 through $n-1$. Prove that A is not invertible.

    Row n is a linear combination of rows 1 through $n-1$ so there are constants $k_1, ..., k_{n-1}$ such that $r_n = k_1 r_1 + k_2 r_2 + ... + k_{n-1} r_{n-1}$ (where $r_1, r_2, ..., r_n$ are the rows of A).

    If we do the following type three row operations $r_n - k_1 r_1 \to r_n, r_n - k_2 r_2 \to r_n, ..., r_n - k_{n-1} r_{n-1} \to r_n$, then the resulting matrix has an n-th row which consists of all zeros. The determinant was not changed by the type three row operations and any matrix with a row of zeros has determinant 0, so $\det(A) = 0$ and A is not invertible.

11. Let A be a 4×4 matrix. Suppose that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. What is $\det(A)$?

    A is a square matrix, so we can use the theorem which lists the conditions which are equivalent to A being invertible. If A was invertible, then $A\mathbf{x} = \mathbf{0}$ would have only the trivial solution. As it has a nontrivial solution, A cannot be invertible so $\det(A) = 0$.

12. Suppose A is a 3×3 matrix with $\det(A) = 6$. Compute the determinant of the following matrices.

    (a) A^3
    $\det(A^3) = \det(A)^3 = 216$

    (b) $2A$
    $\det(2A) = 2^3 \det(A) = (8)(6) = 48$

    (c) $(A^T)^{-1}$
    $\det((A^T)^{-1}) = 1/\det(A^T) = 1/\det(A) = 1/6$.

13. Suppose A and B are invertible 3×3 matrices and $AB^T = 2B^2$. If $\det(A) = 5$, what is $\det(B)$?

$\det(AB^T) = \det(2B^2)$. The left side is $\det(AB^T) = \det(A)\det(B^T) = \det(A)\det(B) = 5\det(B)$. The right side is $\det(2B^2) = 2^3\det(B^2) = 8\det(B)\det(B) = 8\det(B)^2$. This gives us the equation $5\det(B) = 8\det(B)^2$. There are two possible solutions to this equation, $\det(B) = 5/8$ or $\det(B) = 0$, but B is invertible so $\det(B) \neq 0$ and thus $\det(B) = 5/8$.

14. Compute the determinant of A.

 (a) $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

 For 2×2 matrixes, the determinant is $a_{11}a_{22} - a_{12}a_{21}$ so $\det(A) = (3)(5) - (-1)(2) = 17$.

 (b) $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

 There are a lot of different methods that can be used to find this determinant. For review purposes we will go over all of them. On an exam, you can use whatever method you find easiest. Time permitting, you may want to use more than one method as a check.

 Method 1: Using the definition of determinant. There are two ways to pick 3 nonzero entries so that we have exactly one from each row and column. We can take the 5 from column 1, the 1 from column 2, and the 3 from column 3 or we can take the 5 from column 1, the -1 from column 2, and the -2 from column 3. The first choice of three entries corresponds to the permutation 213 which has 1 inversion so is odd. The second choice corresponds to the permutation 312 which has 2 inversions so is even. The determinant is therefore $-(5)(1)(3) + (5)(-1)(-2) = -5$.

 Method 2: Reduction to triangular form. The row operations $r_1 \leftrightarrow r_2$, $r_3 + r_2 \to r_3$ will give you the upper triangular matrix $\begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. The determinant of an upper triangular matrix is the product of its diagonal entries so this matrix has determinant $(5)(1)(1) = 5$. The first row operation was type 1 so it swapped the sign of the determinant and the second was type 3 so it does not change the determinant so the determinant of the original matrix A is -5.

 Method 3: Cofactor Expansion. Using cofactor expansion along the first column, we get that $\det(A) = -5\det\left(\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\right) = -5((1)(3) -$

$(-2)(-1)) = -5$.

Method 4: We can also use the trick for 3×3 matrices where we repeat the first and second column next to A. See Example 8 in Section 3.1 for a more detailed explanation of this method.

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 2 & 0 & 0 & 9 \end{bmatrix}$

Method 1: Using the definition of determinant. We are looking for ways to pick out 4 nonzero entries so that we have exactly one from each row and each column. In column 2 we must take the second entry. In column 3, we cannot take the second entry since we already have something from row 2 so we must take the 3rd entry. Then we can either take the first entry from the first column and fourth entry from the fourth column, or we can take the fourth entry from column 1 and the first entry from column 4. We see that the determinant has 2 nonzero terms which are $a_{11}a_{22}a_{33}a_{44} = (1)(3)(5)(9) = 135$ and $a_{14}a_{22}a_{33}a_{41} = (6)(3)(5)(2) = 180$. We also need to determine the sign $\pm$ to go along with each term. The first term corresponds to the permutation 1234 which has no inversions so is even and gets a $+$. The second term corresponds to 4231 which has inversions $42, 43, 41, 21, 31$ so it has 5 inversions and is odd and gets a $-$. Hence $\det(A) = 135 - 180 = -45$.

Method 2: Reduction to triangular form. The single row operation $r_4 - 2r_1 \to r_4$ will result in the upper triangular matrix $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. The determinant of an upper triangular matrix is the product of the diagonal entries so this matrix has determinant $(1)(3)(5)(-3) = -45$. As the row operation we did was a type 3 row operation, it does not change the determinant and hence $\det(A)$ is also -45.

Method 3: Cofactor expansion. Using cofactor expansion along the second column we get $\det(A) = 3 \det\left(\begin{bmatrix} 1 & 0 & 6 \\ 0 & 5 & 8 \\ 2 & 0 & 9 \end{bmatrix}\right)$. Expanding along the first row, this is $3\left[1 \det\left(\begin{bmatrix} 5 & 8 \\ 0 & 9 \end{bmatrix}\right) + 6 \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}\right)\right] = 3(45 + -60) = 3(-15) = -45$.

Of course, you can also use a combination of methods. For example, you could use cofactor expansion to get down to 3×3 matrices then use the trick for 3×3 matrices to compute the 3×3 determinants.

15. The matrix $A = \begin{bmatrix} 1 & 2 & 6 & 8 \\ 1 & 3 & 0 & 9 \\ 1 & 4 & 0 & 10 \\ 1 & 5 & 7 & 0 \end{bmatrix}$ is invertible. Find all solutions to the following linear systems.

(a) $A^{-1}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

If we multiply $A^{-1}\mathbf{x} = \mathbf{b}$ by A, we get $\mathbf{x} = A\mathbf{b}$ is the only solution. This is $A\mathbf{b} = \begin{bmatrix} -2 \\ 5 \\ 6 \\ 0 \end{bmatrix}$.

(b) $A\mathbf{x} = \mathbf{0}$

A is invertible so the only solution is the trivial solution, $\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.