Quiz 3 Solutions

1. Let $A = \begin{bmatrix} 9 & 4 \\ -4 & -1 \end{bmatrix}$. Find the eigenvalues of A. (6 pts)

   The characteristic polynomial of A is $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 9 & -4 \\ 4 & \lambda + 1 \end{bmatrix}\right) = (\lambda - 9)(\lambda + 1) + 16 = \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1)$. This is 0 when λ is 1 or 7 so the eigenvalues of A are 1 and 7.

2. Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{bmatrix}$. One of the eigenvalues of A is 5. Find a basis for the eigenspace associated with the eigenvalue 5. (7 pts)

   The eigenspace associated with 5 is the same as the null space of $5I - A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. This matrix has RREF $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The null space is all vectors of the form $\begin{bmatrix} z \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ 0 & -3 & 1 \end{bmatrix}$. For what value (if any) of c is the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ c \end{bmatrix}$ an eigenvector of A? Either find c and the associated eigenvalue, or explain why no such c exists. (7 pts)

   We want to find λ such that $A\mathbf{v} = \lambda\mathbf{v}$. $A\mathbf{v} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ c \end{bmatrix} = \begin{bmatrix} 1 + c \\ -5 + c \\ 3 + c \end{bmatrix}$ and $\lambda\mathbf{v} = \begin{bmatrix} 2\lambda \\ -\lambda \\ c\lambda \end{bmatrix}$ so we get the equations $2\lambda = 1 + c$, $-\lambda = -5 + c$, and $c\lambda = 3 + c$. The first two equations $2\lambda = 1 + c$ and $-\lambda = -5 + c$ are linear in c and λ and have unique solution $c = 3$ and $\lambda = 2$. This works in the third equation $c\lambda = 3 + c$, so $\mathbf{v}$ is an eigenvector when $c = 3$ and the associated eigenvalue is $\lambda = 2$.

Bonus: Determine if the following is true or false. Give a proof or counterexample. (5 pts)
If A has eigenvalue λ, then $A + A^T$ must have eigenvalue 2λ.

False. One possible counterexample is the matrix $A = \begin{bmatrix} 9 & 4 \\ -4 & -1 \end{bmatrix}$ from problem 1. A has eigenvalues 1 and 7 and $A + A^T = \begin{bmatrix} 18 & 0 \\ 0 & -2 \end{bmatrix}$ which has eigenvalues 18 and -2, which are not 2 times the eigenvalues of A.