Quiz 2 Solutions

1. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$.

   (a) Find the lengths of the vectors in S. (2 pts)

   These vectors are all length 1. Use that the length of $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\sqrt{x^2 + y^2}$.

   (b) Is the set S orthogonal, orthonormal, or neither? Explain. (4 pts)

   This set is neither. Not all pairs of vectors from S are orthogonal. For example the first vector and the third vector are not orthogonal because their dot product is $1/\sqrt{2}$, not 0. As it is not orthogonal, it is also not orthonormal.

2. Find an orthogonal basis for the 2-dimensional subspace of $\mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

   (6 pts)

   Let $\mathbf{u_1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and let $\{\mathbf{v_1}, \mathbf{v_2}\}$ be the orthogonal basis we're trying to construct. Use the Gram-Schmidt formula to find the $\mathbf{v_i}$.

   The first vector is $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

   The second is $\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{v_1} \cdot \mathbf{u_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

   The orthogonal basis we get is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3/2 \\ 5/2 \end{bmatrix} \right\}$. Another possible answer is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\}$.

3. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. (4 points each)

Question 1: What can we say about n?

$n \geq 3$. Any orthogonal set of nonzero vectors is linearly independent. S is orthogonal and the vectors are nonzero because they have length 1, so S is a linearly independent set in $\mathbb{R}^n$. The dimension of $\mathbb{R}^n$ is n so it cannot contain a linearly independent set of size larger than n, so n must be a least 3.

Question 2: Let $\mathbf{u} = \mathbf{v_1} - \mathbf{v_3}$ and $\mathbf{w} = 3\mathbf{v_1} + 2\mathbf{v_2}$. What is $\mathbf{u} \cdot \mathbf{w}$?

$\mathbf{u} \cdot \mathbf{w} = 3$. Using properties of dot products,

$$\mathbf{u} \cdot \mathbf{w} = (\mathbf{v_1} - \mathbf{v_3}) \cdot (3\mathbf{v_1} + 2\mathbf{v_2})$$

$$= 3(\mathbf{v_1} \cdot \mathbf{v_1}) - 3(\mathbf{v_3} \cdot \mathbf{v_1}) + 2(\mathbf{v_1} \cdot \mathbf{v_2}) - 2(\mathbf{v_3} \cdot \mathbf{v_2}) \ .$$

As S is orthogonal, $\mathbf{v_3} \cdot \mathbf{v_1} = \mathbf{v_1} \cdot \mathbf{v_2} = \mathbf{v_3} \cdot \mathbf{v_2} = 0$. Also, the vectors in S have length 1 so $\mathbf{v_1} \cdot \mathbf{v_1} = \|\mathbf{v_1}\|^2 = 1^2 = 1$. Plugging in these values gives

$$\mathbf{u} \cdot \mathbf{w} = 3(1) - 3(0) + 2(0) - 2(0) = 3 \ .$$